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Full Paper

Sufficiency and duality of multi-objective bi-level programming problem under Guignard constraint qualification

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Abstract: Although a great deal of work has been done on bi-level programming problems (BPPs) during the last few decades, there is still a substantial gap in the literature on sufficient optimality conditions and duality results of BPPs. This article aims to develop sufficient optimality conditions and Mond-Weir-type dual for a multi-objective BPP from an optimistic perspective under the Guignard constraint qualification. Using the weighted sum scalarisation at the lower level and a special function at the upper level, we convert the multi-objective BPP to the corresponding single-objective BPP. With the help of optimal value reformulation, we reformulate the single-objective BPP as a single-level mathematical programming problem. Using the generalised convexity, we develop sufficient optimality conditions. Moreover, we formulate Mond-Weir type dual and establish the corresponding duality results.

Keywords: bi-level programming, sufficient optimality conditions, Clarke sub-differential, convexifactors, Mond-Weir dual

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INTRODUCTION

Bi-level programming problems (BPPs) have gained popularity in the field of optimisation theory owing to their enormous real-life applications, for example in transportation, economics, management and engineering. The nested structure of BPPs distinguishes them from other optimisation problems. The BPPs are represented by a leader-follower structure, in which the leader tries to optimise his/her problem subject to the follower's response.

Many authors have been exploring BPPs over the last few years. Bard [1] developed the first-order necessary optimality conditions under the differentiability assumption. Using differential stability results for the parametric optimisation problem, Dempe [2] obtained the necessary and sufficient conditions for the local optimal solutions of the BPPs. Babahadda and Gadhi [3] established the necessary optimality conditions for the bi-level optimisation problem in terms of convexifactors. Lafhim et al. [4] used Ψ-reformulation to develop necessary optimality conditions for a multi-objective BPP. Chuong [5] established Fritz-John and Karush-Kuhn-Tucker necessary conditions for a non-smooth multi-objective bi-level optimisation problem having vector-valued functions in both levels. Mehlitz and Zemkoho [6] considered an optimistic bi-level optimisation problem involving smooth functions to study the first-order and second-order sufficient optimality conditions. In terms of limiting sub-differentials and limiting normal cones, Dempe et al. [7] derived necessary optimality conditions for a non-smooth semi-vectorial bi-level optimisation. Despite extensive research on the necessary optimality conditions of BPPs, the literature is scarce on sufficient conditions, which prompted us to develop sufficient optimality conditions for the multi-objective BPP. Furthermore, we construct a Mond-Weir-type dual corresponding to the considered problem and establish the duality results.

In this article we consider a bi-level problem with multiple objectives at both levels. We employ some techniques to convert these vector optimisation problems to scalar optimisation problems so that we can use the previously established results of the single-objective case to write optimality conditions for the multi-objective optimisation case. We use the widely utilised weighted sum scalarisation for the lower level, a special function introduced by Hiriart-Urruty [8] for the upper level, and then the well-known optimal-value reformulation to convert the BPP to a single-level problem.

The single-level problem so obtained is non-smooth. As a result, by using non-smooth analysis tools, we derive sufficient optimality conditions. For non-smooth functions, generalised sub-differentials provide sharp extremality conditions as well as good calculus rules. Demyanov [9] proposed a convex and compact convexificator in 1994. The term convexifactor was coined by Dutta and Chandra [10] to refer to both convexificators and non-convex convexificators. The concept of convexifactor is a generalisation of some well-known sub-differentials such as Clarke sub-differential [11], Michel and Penot sub-differential [12] and Mordukhovich sub-differential [13]. Here we present sufficient optimality conditions in terms of convexifactors.

The principle of duality in mathematical optimisation theory states that optimisation problems can be viewed from one of two aspects, the primal problem or the dual problem. Suneja and Kohli [14] derived sufficient optimality conditions for the BPP using convexifactors and established various duality results for Wolfe-type dual and Mond-Weir- type dual. Gadhi et al. [15] established sufficient optimality conditions, formulated the Mond-Weir-type dual and provided duality results for a bi-level problem with multiple objectives at the upper level. Recently Van Su et al. [16] considered a non-smooth multi-objective BPP with equilibrium constraints and studied a Wolfe- and Mond-Weir-type dual problem corresponding to it. Due to the lack of literature on the duality theory on BPPs, we further develop a Mond-Weir-type dual for a multi-objective bi-level problem and establish weak and strong duality results.

PRELIMINARIES

Let R^n_+ be the cone of all non-negative vectors in R^n and $e \in R^n$ be all-ones vector such that $e = \sum_{i=1}^{n} e_i$. For a subset P of R^n, the sets $cl\ P, cone\ P, cl\ cone\ P$ and P^- notify the closure of P, the convex cone generated by P, the closed convex cone generated by P and the negative polar cone of P respectively.

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Let P be a non-empty subset of R^n and consider the functions $d_p: R^n \to [0, +\infty)$ defined as

$$d_p(x) = \inf_{y \in P} \|x - y\|, \quad \forall x \in R^n.$$

Hiriart-Urruty [8] introduced the following function: for $P \subset R^n$, consider the function

$$\Delta_P(x) = \begin{cases} -d_{(R^n \setminus P)}(x) & if\ x \in P, \\ d_P(x) & if\ x \in R^n \setminus P. \end{cases}$$

Let P be a subset of R^n and $x \in cl\,P$. The contingent cone $T(P, x)$ to P at x is defined as

$$T(P, x) = \{v \in R^n : \exists\ t_n \downarrow 0 \text{ and } \exists v_n \to v \text{ such that } x + t_n v_n \in P, \forall n\}.$$

A function $f: R^n \to \mathrm{R} \cup \{+\infty\}$ is said to be locally Lipschitzian around $\bar{x} \in \mathrm{dom}\, f$ if there is a neighbourhood W of $\bar{x}$ and $k \geq 0$ such that

$$|f(x) - f(y)| \leq k\|x - y\|, \quad \forall x, y \in W.$$

When f is locally Lipschitz, the generalised directional derivative

$$v \to f^0(\bar{x}, v) = \lim_{x \to \bar{x}} \sup_{t \to 0^+} \frac{f(x + tv) - f(x)}{t}$$

is a finite sublinear function. The following set

$$\partial^c f(\bar{x}) = \{x^* \in R^n : \langle x^*, v \rangle \leq f^0(\bar{x}, v), \quad \forall v \in R^n\},$$

representing the Clarke sub-differential of f at $\bar{x}$, is a non-empty convex compact subset of R^n.

Let $F: R^n \to R \cup \{+\infty\}$ and let $x \in R^n$ where $F(x)$ is finite. Then the upper and lower Dini directional derivatives of F at x in the direction v are defined respectively as

$$F_d^-(x, v) = \liminf_{t \to 0^+} \frac{F(x + tv) - F(x)}{t}$$

and

$$F_d^+(x, v) = \limsup_{t \to 0^+} \frac{F(x + tv) - F(x)}{t}.$$

Definition 1 [10, 17]**.** Let $F: R^n \to R \cup \{+\infty\}$ be a function.

- The function F is said to admit an upper convexifactor (UCF) $\partial^* F(x)$ at x if $\partial^* F(x) \subset R^n$ is closed and for each $v \in R^n$,
$$F_d^-(x, v) \leq \inf_{x^* \in \partial^* F(x)} \langle x^*, v \rangle.$$
- The function F is said to admit a lower convexifactor $\partial_* F(x)$ at x if $\partial_* F(x) \subset R^n$ is closed and for each $v \in R^n$,
$$F_d^+(x, v) \geq \inf_{x^* \in \partial_* F(x)} \langle x^*, v \rangle.$$
- F is said to admit a convexifactor $\partial^* F(x)$ at x if $\partial^* F(x)$ is both an upper and lower convexifactor of F at x.
- F is said to have an upper semi-regular convexifactor (USRCF) $\partial^* F(x)$ at x if $\partial^* F(x)$ is an upper convexifactor at x and for each $v \in R^n$,
$$F_d^+(x, v) \leq \sup_{x^* \in \partial^* F(x)} \langle x^*, v \rangle.$$

Definition 2 [18]. Let $F: R^{n_1} \times R^{n_2} \to R$ and $(u, v) \in R^{n_1} \times R^{n_2}$. Assuming that F admits convexifactor $\partial^* F(u, v)$, F is said to be

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- ∂^*-quasi-convex at (u,v) if and only if for all $(x,y) \in R^{n_1} \times R^{n_2}$,
$$F(x,y) - F(u,v) \leq 0 \Rightarrow \langle \rho, (x,y) - (u,v) \rangle \leq 0, \ \forall \rho \in \partial^* F(u,v);$$
- ∂^*-pseudo-convex at (u,v) if and only if for all $(x,y) \in R^{n_1} \times R^{n_2}$,
$$F(x,y) - F(u,v) < 0 \Rightarrow \langle \rho, (x,y) - (u,v) \rangle < 0, \ \forall \rho \in \partial^* F(u,v).$$

Definition 3 [19]. Let Ω be a non-empty convex subset of R^n. A local Lipschitz function $F : \Omega \to R$ is said to be generalised quasi-convex (∂^c-quasi-convex) if $\forall\, x, y \in \Omega$ and for all $\xi \in \partial^c F(x)$, we have

$$F(y) \leq F(x) \Rightarrow \langle \xi, y - x \rangle \leq 0.$$

A feasible point $(\tilde{x}, \tilde{y})$ is said to be a weak efficient solution of (Q) if

$$F(x,y) - F(\tilde{x},\tilde{y}) \notin -\text{int}\, R^n_+, \qquad \forall (x,y) \in C.$$

PROBLEM STATEMENT

In this section we consider a multi-objective BPP (Q). With the help of scalarisation and optimal value function reformulation we transform (Q) to a single-level mathematical programming problem:

$$(Q): \begin{cases} R^n_+ - \min\limits_{x,y} F(x,y) = (F_1(x,y), \dots, F_n(x,y)) \\ \quad \text{s.t.} \quad G_j(x,y) \leq 0, \qquad \forall\, j \in J, \\ \qquad (x,y) \in R^{n_1} \times R^{n_2}, y \in \Upsilon_{\text{weff}}(x), \end{cases}$$

where, for each $x \in R^{n_1}$, $\Upsilon_{\text{weff}}(x)$ is the weak efficient solution set of the following multi-objective parametric optimisation problem:

$$(Q_x): \begin{cases} R^m_+ - \min\limits_{y} f(x,y) = (f_1(x,y), \dots, f_m(x,y)) \\ \quad s.t. \quad g_i(x,y) \leq 0, \qquad \forall\, i \in I, \\ \qquad (x,y) \in R^{n_1} \times R^{n_2}. \end{cases}$$

Here F_k, $G_j: R^{n_1} \times R^{n_2} \to R$ are local Lipschitz functions, $\forall k \in K = \{1, \dots, n\}$, $\forall\, j \in J = \{1, \dots, p\}$ and g_i, $f_s: R^{n_1} \times R^{n_2} \to R$ are given convex continuous functions, $\forall\, i \in I = \{1, \dots, q\}, \forall\, s \in S = \{1, \dots, m\}$; $m, n, n_1, n_2, p, q \geq 1$ are integers.

For a fixed $x \in R^{n_1}$, let

$$\Psi(x) = \{y \in R^{n_2} : g_i(x,y) \leq 0, \quad \forall i \in I\},$$

where $y \in \Psi(x)$ is said to be a local weak efficient solution of (Q_x). Let

$$C = \{(x,y) \in R^{n_1} \times R^{n_2} \colon y \in \Upsilon_{weff}(x), G_j(x,y) \leq 0, \forall\, j \in J\}$$

represent the feasible set of (Q).

REFORMULATION

Let $S = \{z \in R^m_+$ such that $z^T e = 1\}$. Now for any $z \in S$, we consider the following scalarised problem:

$$(Q_{zx}): \begin{cases} \min\limits_{y} z^T f(x,y) \\ \text{s.t.} \quad y \in \Psi(x). \end{cases}$$

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Due to convexity of the set $A_x = \{f(x,y)\colon\ y \in \Psi(x)\} + R_+^m$, for any $x \in R^{n_1}$, by Jahn [20] we have

$$\Upsilon_{weff}(x) = \cup_{z\in S}\Theta(z,x),$$

Where $\Theta\colon S \times R^{n_1} \to R^{n_2}$ is the solution set-valued mapping of (Q_{zx}) defined by

$$\Theta(z,x) = \{y \in \Psi(x) \text{ such that } z^T f(x,y) \le \Lambda(z,x)\},$$

and $\Lambda\colon S \times R^{n_1} \to R$ characterised by

$$\Lambda(z,x) = \inf_{y\in\Psi(x)} z^T f(x,y)$$

is the optimal value function of (Q_{zx}).

Throughout the article, the set-valued mapping Θ is assumed to be closed at all points $(\bar{z},\bar{x})$ where $\bar{z} \in S$. To obtain the Lipschitz continuity of the scalarised problem's value function, we need the inner semi-continuity of the new lower-level problem's solution set-valued mapping.

Given $(\bar{x},\bar{y}) \in C$ and $\bar{z} \in S$ such that $\bar{y} \in \Theta(\bar{z},\bar{x})$, the solution set-valued mapping Θ is said to be inner semi-continuous at $(\bar{z},\bar{x},\bar{y})$ if and only if for every sequence $x_k \to \bar{x}$ and $z_k \to \bar{z}$ there is a sequence $y_k \in \Theta(z_k,x_k)$ converging to $\bar{y}$ as $k \to \infty$.

Lemma 1. Let $(\bar{x},\bar{y}) \in C$ be a global weak efficient solution of (Q) and let $\bar{z} \in S$ such that $\bar{y} \in \Lambda(\bar{z},\bar{x})$. Then $(\bar{z},\bar{x},\bar{y})$ is a global optimal solution of

$$(\mathrm{Q_e})\colon \begin{cases} \min\limits_{z,x,y} \Delta_{-\mathrm{int}\,\mathrm{R}_+^{\mathrm{n}}}\big(\mathrm{F(x,y)} - \mathrm{F(\bar{x},\,\bar{y})}\big) \\ \text{s.t.} \quad G_j(x,y) \le 0, \quad j = 1,\dots,p, \\ \qquad\; g_i(x,y) \le 0, \quad i = 1,\dots,q, \\ \qquad\; z^T f(x,y) - \Lambda(z,x) \le 0, \\ \qquad\; z_l \ge 0, \quad l = 1,\dots,m, \ \ z^T e = 1, \end{cases}$$

Proof. The proof follows along the line of Lemma 1 proposed by Dempe et al. [21].

Definition 4 [21]**.** The non-smooth generalised Guignard constraint qualification (GCQ) holds at $(\bar{x},\bar{y})$ if $\forall\ \bar{z} \in S$ such that $\bar{y} \in \Theta(\bar{z},\bar{x})$. Then one has

$$\left[T\big(\Gamma,(\bar{z},\bar{x},\bar{y})\big)\right]^- \subseteq \mathrm{cl}\big(\textstyle\sum_{i\in\Pi(\bar{z},\bar{x},\bar{y})} cone\ \partial^*\varphi_i(\bar{z},\bar{x},\bar{y})\big),$$

where

$$\begin{aligned} &\varphi_0(\bar{z},\bar{x},\bar{y}) = \bar{z}_k f(\bar{x},\bar{y}) - \Lambda(\bar{z}_k,\bar{x}), \\ &\varphi_j(\bar{z},\bar{x},\bar{y}) = G_j(\bar{x},\bar{y}), \quad j = 1,\dots,p, \\ &\varphi_i(\bar{z},\bar{x},\bar{y}) = g_i(\bar{x},\bar{y}), \quad i = 1,\dots,q, \\ &\varphi_k(\bar{z},\bar{x},\bar{y}) = -\bar{z}_k, \qquad k = 1,\dots,l, \\ &\varphi_{p+q+l+1}(\bar{z},\bar{x},\bar{y}) = \bar{z}^T e - 1, \\ &\varphi_{p+q+l+2}(\bar{z},\bar{x},\bar{y}) = -\bar{z}^T e + 1, \end{aligned}$$

$$\Pi(\bar{z},\bar{x},\bar{y}) = \{i \in \theta\colon \varphi_i(\bar{z},\bar{x},\bar{y}) = 0\}, \ \ \theta = \{0,1,\dots,p+q+l+2\}$$

and Γ is the feasible set of (Q_e) defined by

$$\Gamma = \{(\bar{z},\bar{x},\bar{y}) \in R^m \times R^{n_1} \times R^{n_2}\colon \varphi_i(\bar{z},\bar{x},\bar{y}) \le 0, \ \ i \in \theta\}.$$

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SUFFICIENT OPTIMALITY CONDITION

In this section we present the sufficient optimality condition for (Q) by using the optimistic approach. We now state the following regularity condition, which are used to obtain the optimality and duality results.

Let

$$I_0(x,y) = \{i \in I: g_i(x,y) = 0\},$$
$$J_0(x,y) = \{j \in J: G_j(x,y) = 0\}$$

and

$$I_0^{\neq}(x,y) = \{i \in I: g_i(x,y) \neq 0\},$$
$$J_0^{\neq}(x,y) = \{j \in J: G_j(x,y) \neq 0\}.$$

We say that (Q_{zx}) is lower-level regular at $(\bar{x}, y) \in R^{n_1} \times R^{n_2}, y \in \Theta(\bar{z}, \bar{x})$ if

$$\left\{ \sum_{i \in I_0(\bar{x},y)} \mu_i v_i = 0, \quad \mu_i \geq 0 \right\} \Rightarrow \{\mu_i = 0, \ \forall i \in I_0(\bar{x}, y)\}$$

whenever $v_i \in \partial^c g_i(\bar{x}, y)$ with some $u_i \in R^{n_1}$ as $i \in I_0(\bar{x}, y)$.

The following necessary optimality condition for (Q) (Theorem 1) has been proved by Dempe et al. [21].

Theorem 1 (Necessary condition). Let $(\bar{x}, \bar{y}) \in C$ be a local weak efficient solution of (Q) and let $\bar{z} \in S$ such that $\bar{y} \in \Theta(\bar{z}, \bar{x})$. Assume that $F_k, k \in K$ admit bounded USRCF $\partial^* F_k(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y})$ and that $G_j, \ j \in J, \ g_i, \ i \in I$ admit UCFs $\partial^* G_j(\bar{x}, \bar{y}), \ \partial^* g_i(\bar{x}, \bar{y})$ respectively at $(\bar{x}, \bar{y})$. Suppose that the non-smooth generalised GCQ holds at $(\bar{z}, \bar{x}, \bar{y})$, that the solution-set mapping Θ is inner semicontinuous at $(\bar{z}, \bar{x}, \bar{y})$, and that for each vector $y \in \Theta(\bar{z}, \bar{x})$, (Q_{zx}) is lower-level regular at $(\bar{x}, y)$. Also, suppose that $\overline{\text{cone}}(A \cup B) \subset \overline{\text{cone}}A + \overline{\text{cone}}B$. Then there exists $\bar{\pi}^* = (\lambda^*, \rho^*, \sigma^*, \tau^*) \in R_+^{n+p+q+1}, \lambda^* \neq 0_{R^n}$ such that

$$\begin{aligned} 0 \in \sum_{k=1}^{n} \{0\} \times \lambda_k^* \, \partial^* F_k(\bar{x}, \bar{y}) + \sum_{j=1}^{p} \{0\} \times \rho_j^* \, \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i=1}^{q} \{0\} \times \sigma_i^* \, \partial^* g_i(\bar{x}, \bar{y}) \\ + \tau^* \partial^c \big((\bar{z}^T f)(\bar{z}, \bar{x}, \bar{y}) - \Lambda(\bar{z}, \bar{x}) \times \{0\} \big), \\ \rho_j^* G_j(\bar{x}, \bar{y}) = 0, \ \ \sigma_i^* g_i(\bar{x}, \bar{y}) = 0, \ \ j \in J, \ \ i \in I, \end{aligned} \tag{1}$$

where ∂^c stands for the Clarke sub-differential.

If the variable z is fixed to $\bar{z}$, we have the following sufficient condition.

Theorem 2 (Sufficient condition). Let $\bar{u} = (\bar{x}, \bar{y}) \in C$ be a feasible solution of (Q) and let $\bar{z} \in S$ such that $\bar{y} \in \Theta(\bar{z}, \bar{x})$. Suppose that $F_k, k \in K$ is ∂^*-pseudo-convex at $(\bar{x}, \bar{y})$ and $G_j, \ j \in J_0(\bar{u}), g_i, i \in I_0(\bar{u})$ are ∂^*-quasi-convex at $(\bar{x}, \bar{y})$ and $(\bar{z}^T f)(\bar{z}, \bar{x}, \bar{y}) - \Lambda(\bar{z}, \bar{x}) \times \{0\}$ is ∂^c-quasi-convex at $(\bar{z}, \bar{x}, \bar{y})$ and there exists $\bar{\pi}^* = (\lambda^*, \rho^*, \sigma^*, \tau^*) \in R_+^{n+p+q+1}, \lambda^* \neq 0_{R^n}$ such that (1) holds. Then $\bar{u}$ is a weak efficient solution of (Q).

Proof. To the contrary, suppose that $\bar{u}$ is not a weak efficient solution of (Q). Then there exists $u = (\tilde{x}, \tilde{y}) \in C$ such that

$$F(u) - F(\bar{u}) \in -\text{int} \, R_+^n.$$

Then $F_k(u) - F_k(\bar{u}) < 0, \ \forall \, k \in K$.

By (1), we obtain $\xi_k \in \partial^* F_k(\bar{u})$, $\mu_j \in \partial^* G_j(\bar{u})$, $j \in J$, $\nu_i \in \partial^* g_i(\bar{u})$, $i \in I$ and $\delta \in \partial^c((\bar{z}^T f)(\bar{z}, \bar{x}, \bar{y}) - \Lambda(\bar{z}, \bar{x}) \times \{0\})$ such that

$$0 = \sum_{k=1}^{n} \{0\} \times \lambda_k^* \xi_k + \sum_{j=1}^{p} \{0\} \times \rho_j^* \mu_j + \sum_{i=1}^{q} \{0\} \times \sigma_i^* \nu_i + \tau^* \delta,$$

and

$$\rho_j^* G_j(\bar{u}) = 0, \;\; \sigma_i^* g_i(\bar{u}) = 0, \qquad j \in J, \;\; i \in I.$$

Since F_k, $k \in K$ is ∂^*-pseudo-convex at $\bar{u}$, we have

$$\langle \xi_k, u - \bar{u} \rangle < 0, \qquad \forall\, k \in K.$$

As $\lambda^* \in R_+^n \setminus \{0\}$, it follows that

$$\langle \lambda_k^* \xi_k, u - \bar{u} \rangle < 0, \qquad \forall\, k \in K.$$

Therefore,

$$\langle \sum_{j=1}^{p} \{0\} \times \rho_j^* \mu_j + \sum_{i=1}^{q} \{0\} \times \sigma_i^* \nu_i + \tau^* \delta, \;\; \{0\} \times (u - \bar{u}) \rangle > 0. \tag{2}$$

Since $u \in C$, we have

$$G_j(u) \le 0, g_i(u) \le 0, (z^T f)(z, \tilde{x}, \tilde{y}) \le \Lambda(z, \tilde{x}) \qquad \forall\, j \in J, \qquad \forall\, i \in I.$$

On the one hand, in the case $j \in J_0^{\neq}(\bar{u})$ we get $\rho_j^* = 0$; in the case $i \in I_0^{\neq}(\bar{u})$ we get $\sigma_i^* = 0$. Therefore,

$$\begin{aligned} \langle \rho_j^* \mu_j, u - \bar{u} \rangle &= 0, \qquad \forall\, j \in J_0^{\neq}(\bar{u}), \\ \langle \sigma_i^* \nu_i, u - \bar{u} \rangle &= 0, \qquad \forall\, i \in I_0^{\neq}(\bar{u}). \end{aligned} \tag{3}$$

On the other hand, since $(\bar{z}^T f)(\bar{z}, \bar{x}, \bar{y}) - \Lambda(\bar{z}, \bar{x}) = 0$ and since $G_j(\bar{u}) = 0 = g_i(\bar{u})$, $\forall j \in J_0(\bar{u})$, $\forall i \in I_0(\bar{u})$, we have

$$\begin{aligned} G_j(u) - G_j(\bar{u}) &\le 0, \qquad \forall j \in J_0(\bar{u}), \\ g_i(u) - g_i(\bar{u}) &\le 0, \qquad \forall\, i \in I_0(\bar{u}), \\ [(z^T f)(z, \tilde{x}, \tilde{y}) - \Lambda(z, \tilde{x})] - [(\bar{z}^T f)(\bar{z}, \bar{x}, \bar{y}) - \Lambda(\bar{z}, \bar{x})] &\le 0. \end{aligned}$$

Since G_j, g_i are ∂^*-quasi-convex at $\bar{u}$ and $(z^T f)(z, x, y) - \Lambda(z, x)$ is ∂^c-quasi-convex at $(\bar{z}, \bar{u})$, we get

$$\begin{aligned} \langle \mu_j, u - \bar{u} \rangle &\le 0, \qquad \forall\, j \in J_0(\bar{u}), \\ \langle \nu_i, u - \bar{u} \rangle &\le 0, \qquad \forall\, i \in I_0(\bar{u}), \\ \langle \delta, (z - \bar{z}, u - \bar{u}) \rangle &\le 0. \end{aligned}$$

As $\tau^* \ge 0$, $\rho_j^* \ge 0$ and $\sigma_i^* \ge 0$, one gets

$$\begin{aligned} \langle \{0\} \times \rho_j^* \mu_j, \{0\} \times (u - \bar{u}) \rangle &\le 0, \qquad \forall\, j \in J_0(\bar{u}), \\ \langle \{0\} \times \sigma_i^* \nu_i, \{0\} \times (u - \bar{u}) \rangle &\le 0, \qquad \forall\, i \in I_0(\bar{u}), \\ \langle \tau^* \delta, (z - \bar{z}, u - \bar{u}) \rangle &\le 0. \end{aligned} \tag{4}$$

Combining (3) and (4), since $J = J_0^{\neq}(\bar{u}) \cup J_0(\bar{u})$ and $I = I_0^{\neq}(\bar{u}) \cup I_0(\bar{u})$, we obtain

$$\begin{aligned} \langle \{0\} \times \rho_j^* \mu_j, \{0\} \times (u - \bar{u}) \rangle &\le 0, \qquad \forall\, j \in J, \\ \langle \{0\} \times \sigma_i^* \nu_i, \{0\} \times (u - \bar{u}) \rangle &\le 0, \qquad \forall\, i \in I, \\ \langle \tau^* \delta, (z - \bar{z}, u - \bar{u}) \rangle &\le 0. \end{aligned}$$

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Summing these inequalities, we obtain a contradiction with (2). The proof is then complete.

MOND-WEIR DUAL

Optimality conditions and duality results are the backbones of the optimisation programme. In mathematical optimisation theory, the principle of duality states that optimisation problems can be viewed from one of two viewpoints: the primal problem or the dual problem. The Wolfe-type dual and Mond-Weir-type dual are the two widely studied duals in optimisation. The Mond-Weir dual has an advantage over the Wolfe dual for the weaker assumptions used. Here we formulate a Mond-Weir-type dual for a multi-objective BPP.

Suppose that $F_k, k \in K$ admits bounded USRCF $\partial^* F_k(v,w)$ at $(v,w) \in R^{n_1} \times R^{n_2}$ and $G_j, j \in J$, $g_i, i \in I$ admit UCFs $\partial^* G_j(.,.)$ and $\partial^* g_i(.,.)$ at (v,w). Also suppose z is fixed. Let (D) be a Mond-Weir dual of problem (Q) defined as follows.

$$(D): \begin{cases} R^n_+ - \max F(v,w) = (F_1(v,w), \dots, F_n(v,w)) \\ s.t.\ 0 \in \sum_{k=1}^{n} \{0\} \times \lambda_k^* \, \partial^* F_k(v,w) + \sum_{j=1}^{p} \{0\} \times \rho_j^* \, \partial^* G_j(v,w) \\ \quad + \sum_{i=1}^{q} \{0\} \times \sigma_i^* \, \partial^* g_i(v,w) + \tau^* \partial^c((z^T f)(z,v,w) \\ \quad -\Lambda(z,v) \times \{0\}), \\ \tau^*(z^T f)(v,w) \ge \tau^* \Lambda(z,v), \\ \rho_j^* G_j(v,w) \ge 0,\ \ \sigma_i^* g_i(v,w) \ge 0,\ \ j \in J,\ \ i \in I, \\ (\lambda_1^*, \dots, \lambda_n^*) \ne (0, \dots, 0), \\ \pi^* = \left(\lambda_1^*, \dots, \lambda_n^*, \rho_1^*, \dots, \rho_p^*, \sigma_1^*, \dots, \sigma_q^*, \tau^*\right) \ge 0. \end{cases}$$

Let $\tilde{C}$ represent the feasible set of dual.

Theorem 3 (Weak duality). For any feasible point $\hat{u} = (\hat{x}, \hat{y})$ of (Q) and for any feasible point (v, w, π^*) of (D), such that $F_k, k \in K$ is ∂^*-pseudo-convex at (v,w), $G_j, j \in J_0(\hat{u})$, $g_i, i \in I_0(\hat{u})$ are ∂^*-quasi-convex at (v,w), and $(z^T f)(z,.,.) - \Lambda(z,.)$ is ∂^c-quasi-convex at (z,v,w), there exists $k_0 \in K$ such that $F_{k_0}(\hat{x}, \hat{y}) \ge F_{k_0}(v,w)$.

Proof. By contrary, assume that there is a feasible point $\hat{u}$ of (Q) and a feasible point (v, w, π^*) of (D) such that

$$F_k(\hat{x}, \hat{y}) - F_k(v,w) < 0, \qquad \forall\, k \in K.$$

Notice that $(\hat{x}, \hat{y}) \ne (v,w)$. As $(\lambda_1^*, \dots, \lambda_n^*) \ge 0, (\lambda_1^*, \dots, \lambda_n^*) \ne (0, \dots, 0)$, we have

$$\sum_{k=1}^{n} \lambda_k^* (F_k(\hat{x}, \hat{y}) - F_k(v,w)) < 0.$$

Since $(v, w, \pi^*) \in \tilde{C}$, we obtain $\xi_k \in \partial^* F_k(v,w), \mu_j \in \partial^* G_j(v,w), \nu_i \in \partial^* g_i(v,w)$ and $\delta \in \partial^c((z^T f)(z,v,w) - \Lambda(z,v))$ such that

$$\sum_{k=1}^{n} \{0\} \times \lambda_k^* \, \xi_k = -\sum_{j=1}^{p} \{0\} \times \rho_j^* \mu_j - \sum_{i=1}^{q} \{0\} \times \sigma_i^* \nu_i - \tau^* \delta, \tag{5}$$

and

$$\rho_j^* G_j(v,w) \geq 0, \sigma_i^* g_i(v,w) \geq 0, \qquad j \in J, \;\; i \in I,$$
$$\tau^*\big((z^T f)(z,v,w) - \Lambda(z,v)\big) \geq 0.$$

As $(\hat{x},\hat{y}) \in C$, we have

$$\rho_j^* G_j(\hat{x},\hat{y}) \leq \rho_j^* G_j(v,w), \qquad j \in J,$$
$$\sigma_i^* g_i(\hat{x},\hat{y}) \leq \sigma_i^* g_i(v,w), \qquad i \in I,$$
$$\tau^*\big((z^T f)(z,\hat{x},\hat{y}) - \Lambda(z,\hat{x})\big) \leq \tau^*((z^T f)(z,v,w) - \Lambda(z,v)).$$

Since $F_k, k \in K$ is ∂^*-pseudo-convex at (v,w), we get

$$\langle \xi_k, \big((\hat{x},\hat{y}) - (v,w)\big)\rangle < 0, \qquad k \in K.$$

Therefore,

$$\langle \sum_{k=1}^{n} \{0\} \times \lambda_k^* \xi_k \,, \{0\} \times \big((\hat{x},\hat{y}) - (v,w)\big)\rangle < 0. \tag{6}$$

Since $G_j, j \in J$ and $g_i, i \in I$ are ∂^*-quasi-convex at (v,w), we obtain

$$\langle \mu_j, \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0, \qquad \forall\, j \in J_0(\hat{x},\hat{y}),$$
$$\langle \nu_i, \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0, \qquad \forall\, i \in I_0(\hat{x},\hat{y}).$$

Then

$$\langle \rho_j^* \mu_j, \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0, \qquad \forall\, j \in J_0(\hat{x},\hat{y}),$$
$$\langle \sigma_i^* \nu_i, \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0, \qquad \forall\, i \in I_0(\hat{x},\hat{y}).$$

For $j \in J_0^{\neq}(\hat{x},\hat{y})$, we have $\rho_j^* = 0$ and for $i \in I_0^{\neq}(\hat{x},\hat{y})$, we have $\sigma_i^* = 0$. Then

$$\begin{aligned} &\langle \sum_{j=1}^{p} \{0\} \times \rho_j^* \mu_j \,, \{0\} \times \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0, \\ &\langle \sum_{i=1}^{q} \{0\} \times \sigma_i^* \nu_i \,, \{0\} \times \big((\hat{x},\hat{y}) - (v,w)\big)\rangle \leq 0. \end{aligned} \tag{7}$$

Let us prove that

$$\langle \tau^* \delta, \big((z,\hat{x},\hat{y}) - (z,v,w)\big)\rangle \leq 0. \tag{8}$$

Indeed, in the case $\tau^* > 0$, we have $\big((z^T f)(z,\hat{x},\hat{y}) - \Lambda(z,\hat{x})\big) \leq ((z^T f)(z,v,w) - \Lambda(z,v))$. Since $(z^T f)(z,.,.) - \Lambda(z,.)$ is ∂^c-quasi-convex at (z,v,w), we obtain $\langle \delta, \big((z,\hat{x},\hat{y}) - (z,v,w)\big)\rangle \leq 0$. Then

$$\langle \tau^* \delta, \big((z,\hat{x},\hat{y}) - (z,v,w)\big)\rangle \leq 0.$$

In the case $\tau^* = 0$, this implies

$$\langle \tau^* \delta, \big((z,\hat{x},\hat{y}) - (z,v,w)\big)\rangle = 0.$$

From (5), (7) and (8), we get

$$\langle \sum_{k=1}^{n} \{0\} \times \lambda_k^* \xi_k , \big((\hat{x},\hat{y}) - (v,w)\big)\rangle$$

$$= \langle -\sum_{j=1}^{p} \{0\} \times \rho_j^* \mu_j - \sum_{i=1}^{q} \{0\} \times \sigma_i^* \nu_i - \tau^* \delta , \big((z,\hat{x},\hat{y}) - (z,v,w)\big)\rangle \geq 0.$$

Hence we get a contradiction to (6). The proof is then complete.

Theorem 4 (Strong duality). Let $\bar{u} = (\bar{x},\bar{y})$ be a weak efficient solution of (Q) where the non-smooth GCQ holds. Then there exists $\bar{\pi}^* = (\lambda^*, \rho^*, \sigma^*, \tau^*) \in R_+^{n+p+q+1}, \lambda^* \neq 0_{R^n}$ such that $(\bar{u}, \bar{\pi}^*)$ is a feasible point of (D). Moreover, if $F_k, k \in K$ is ∂^*-pseudo-convex at $\bar{u}$ and $G_j, j \in J_0(\bar{u}), g_i, i \in I_0(\bar{u})$ are ∂^*-quasi-convex at $\bar{u}$ and $(\bar{z}^T f)(\bar{z},.,.) - \Lambda(\bar{z},.)$ is ∂^c-quasi-convex at $(\bar{z},\bar{u})$, then $(\bar{u}, \bar{\pi}^*)$ is a weak efficient solution of (D).

Proof. Let $\bar{u}$ be a weak efficient solution of (Q) where the non-smooth GCQ holds. According to Theorem 1, we have $\lambda^* \in (-R_+^n)^- \setminus \{0\}$ and $(\rho^*, \sigma^*, \tau^*) \in R_+^{p+q+1}$ such that

$$0 \in \sum_{k=1}^{n} \{0\} \times \lambda_k^* \partial^* F_k(\bar{x},\bar{y}) + \sum_{j=1}^{p} \{0\} \times \rho_j^* \partial^* G_j(\bar{x},\bar{y}) + \sum_{i=1}^{q} \{0\} \times \sigma_i^* \partial^* g_i(\bar{x},\bar{y})$$
$$+ \tau^* \partial^c ((\bar{z}^T f)(\bar{z},\bar{x},\bar{y}) - \Lambda(\bar{z},\bar{x}) \times \{0\}),$$
$$\rho_j^* G_j(\bar{x},\bar{y}) = 0, \sigma_i^* g_i(\bar{x},\bar{y}) = 0, \quad j \in J, \ i \in I.$$

Since $(\bar{z}^T f)(\bar{z},\bar{x},\bar{y}) - \Lambda(\bar{z},\bar{x}) = 0$, this implies that the point $(\bar{z},\bar{u},\bar{\pi}^*)$ is a feasible point of (D). Now to prove that $(\bar{z},\bar{u},\bar{\pi}^*)$ is a weak efficient solution of (D), on the contrary, let there exists a point $(z_*, u_*, \pi_*^*) \in \tilde{C}$ such that

$$F(\bar{u}) - F(u_*) \in -\mathrm{int}(R_+^n).$$

Since $(\bar{z},\bar{u},\bar{\pi}^*)$ is a feasible point of (D) and $\bar{u}$ is a feasible solution of (Q), by Theorem 3 we have $F(\bar{u}) - F(u_*) \notin -\mathrm{int}(R_+^n)$, a contradiction. The proof is then complete.

CONCLUSIONS

We have presented the sufficient optimality conditions and Mond-Weir dual for a multi-objective bi-level optimisation problem from an optimistic perspective. With the help of scalarisation and a special function, together with the optimal-value reformulation under the GCQ, we present the sufficient optimality conditions in terms of convexifactors. Furthermore, we have constructed a Mond-Weir-type dual and established the corresponding duality results.

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REFERENCES

1. J. F. Bard, "Optimality conditions for the bilevel programming problem", *Nav. Res. Logist. Q.*, **1984**, *31*, 13-26.
2. S. Dempe, "A necessary and a sufficient optimality condition for bilevel programming problems", *Optim.*, **1992**, *25*, 341-354.
3. H. Babahadda and N. Gadhi, "Necessary optimality conditions for bilevel optimization problems using convexificators", *J. Global Optim.*, **2006**, *34*, 535-549.
4. L. Lafhim, N. Gadhi, K. Hamdaoui and F. Rahou, "Necessary optimality conditions for a bilevel multiobjective programming problem via a Ψ-reformulation", *Optim.*, **2018**, *67*, 2179-2189.
5. T. D. Chuong, "Optimality conditions for nonsmooth multiobjective bilevel optimization problems", *Ann. Oper. Res.*, **2020**, *287*, 617-642.
6. P. Mehlitz and A. B. Zemkoho, "Sufficient optimality conditions in bilevel programming", *Math. Oper. Res.*, **2021**, *46*, 1573-1598.
7. S. Dempe, N. Gadhi and M. El Idrissi, "Optimality conditions for a nonsmooth semivectorial bilevel optimization problem", *Numer. Funct. Anal. Optim.*, **2021**, *42*, 298-319.
8. J. B. Hiriart-Urruty, "Tangent cones, generalized gradients and mathematical programming in banach spaces", *Math. Oper. Res.*, **1979**, *4*, 79-97.
9. V. F. Demyanov, "Convexification and concavification of positively homogeneous functions by the same family of linear functions", Technical Report 3.208,802, University of Pisa, **1994**.
10. J. Dutta and S. Chandra, "Convexifactors, generalized convexity, and optimality conditions", *J. Optim. Theory Appl.*, **2002**, *113*, 41-64.
11. F. H. Clarke, "Optimization and Nonsmooth Analysis", Society for Industrial and Applied Mathematics, Philadelphia, **1990**.
12. P. Michel and J. P. Penot, "Sub-differential calculus for Lipschitz and non-Lipschitz functions", *Compt Rendus Acad. Sci. Paris Ser. I Math.*, **1984**, *12*, 269-272 (in French).
13. B. S. Mordukhovich and Y. Shao, "On nonconvex subdifferential calculus in banach spaces", *J. Convex Anal.*, **1995**, *2*, 211-227.
14. S. K. Suneja and B. Kohli, "Optimality and duality results for bilevel programming problem using convexifactors", *J. Optim. Theory Appl.*, **2011**, *150*, 1-19.
15. N. Gadhi, K. Hamdaoui and M. El Idrissi, "Sufficient optimality conditions and duality results for a bilevel multiobjective optimization problem via a Ψ-reformulation", *Optim.*, **2020**, *69*, 681-702.
16. T. Van Su, D. D. Hang and N. C. Dieu, "Optimality conditions and duality in terms of convexificators for multiobjective bilevel programming problem with equilibrium constraints", *Comput. Appl. Math.*, **2021**, *40*, Art.no.37.
17. V. Jeyakumar and D. T. Luc, "Nonsmooth calculus, minimality, and monotonicity of convexificators", *J. Optim. Theory Appl.*, **1999**, *101*, 599-621.
18. J. Dutta and S. Chandra, "Convexifactors, generalized convexity and vector optimization", *Optim.*, **2004**, *53*, 77-94.

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19. Q. H. Ansari, C. S. Lalitha and M. Mehta, "Generalized Convexity, Nonsmooth Variational Inequalities, and Nonsmooth Optimization", CRC Press, New York, **2013**.
20. J. Jahn, "Vector Optimization", Springer, Berlin, **2011**.
21. S. Dempe, N. Gadhi and M. El Idrissi, "Optimality conditions in terms of convexificators for a bilevel multiobjective optimization problem", *Optim.*, **2020**, *69*, 1811-1830.