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Full Paper

On the Jacobsthal-circulant-Hurwitz numbers

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Abstract: In this paper we define the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind by using of the Hurwitz matrices which are obtained from the characteristic polynomials of the Jacobsthal-circulant sequences of the first, second and third kind. Then we obtain miscellaneous properties of these sequences.

Keywords: Jacobsthal-circulant sequences, Hurwitz matrix, Binet formulas

INTRODUCTION

Deveci and Karaduman [1] defined the Jacobsthal-circulant sequences of the first, second and third kind respectively:

$$\mathrm{J}_{n}^{1} = -\mathrm{J}_{n-1}^{1} + \mathrm{J}_{n-2}^{1} - 2\mathrm{J}_{n-3}^{1} \text{ for } n > 3 \text{ where } \mathrm{J}_{1}^{1} = \mathrm{J}_{2}^{1} = 0 \text{ and } \mathrm{J}_{3}^{1} = 1,$$

$$\mathrm{J}_{n}^{2} = \mathrm{J}_{n-2}^{2} - 2\mathrm{J}_{n-3}^{2} - \mathrm{J}_{n-4}^{2} \text{ for } n > 4 \text{ where } \mathrm{J}_{1}^{2} = \mathrm{J}_{2}^{2} = \mathrm{J}_{3}^{2} = 0 \text{ and } \mathrm{J}_{4}^{2} = 1$$

and

$$\mathrm{J}_{n}^{3} = -2\mathrm{J}_{n-3}^{3} - \mathrm{J}_{n-4}^{3} + \mathrm{J}_{n-5}^{3} \text{ for } n > 5 \text{ where } \mathrm{J}_{1}^{3} = \mathrm{J}_{2}^{3} = \mathrm{J}_{3}^{3} = \mathrm{J}_{4}^{3} = 0 \text{ and } \mathrm{J}_{5}^{3} = 1.$$

Let P be an $\mathrm{n}^{\mathrm{th}}-$degree real polynomial given by

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n.$$

Hurwitz [2] defined the Hurwitz matrix $\left[H_n = \left[h_{ij}\right]_{n\times n}\right]$ associated with P:

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

$$H_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_0 & a_2 & a_4 & & & & \vdots & \vdots & \vdots \\ 0 & a_1 & a_3 & & & & \vdots & \vdots & \vdots \\ \vdots & a_0 & a_2 & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_1 & & \ddots & & a_n & \vdots & \vdots \\ \vdots & \vdots & a_0 & & & \ddots & a_{n-1} & 0 & \vdots \\ \vdots & \vdots & 0 & & & & a_{n-2} & a_n & \vdots \\ \vdots & \vdots & \vdots & & & & a_{n-3} & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{bmatrix}.$$

Suppose that the $(n+k)^{\text{th}}$ term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. Kalman [3] derived a number of closed-form formulas for the generalised sequence by the companion matrix method as follows:

$$A = \left[a_{i,j}\right]_{k \times k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument, he obtained

$$A^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

It is well known that Jacobsthal numbers, circulant matrix and Hurwitz matrix appear in modern research in many fields from mathematics, physics, computer science, architecture to nature and art [4-16]. The theory of the Jacobsthal circulant sequences of the first, second and third kind were introduced by Deveci and Karaduman [1]. Some linear recurrence sequences were defined and given their various properties by matrix methods [17-28]. This paper expands the concept to the Jacobsthal-circulant-Hurwitz numbers which are defined by using circulant and Hurwitz matrices. Firstly, we define the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind by using the Hurwitz matrices which are obtained from the characteristic polynomials of the Jacobsthal-circulant sequences of the first, second and third kind. Then we derive the relationships between the generating matrices and the elements of the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind. Furthermore, we obtain the Binet formulas for the Jacobsthal-

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

circulant-Hurwitz sequences of the three kinds by matrix methods. Finally, we give miscellaneous properties of the Jacobsthal-circulant Hurwitz sequences of the three kinds, such as the generating functions, the permanent and determinant representations and the sums.

ON JACOBSTHAL-CIRCULANT-HURWITZ NUMBERS

It is easy to see that the characteristic polynomials of the Jacobsthal-circulant sequences of the first, second and third kind respectively:

$$f^{(1)}(x)=x^3+x^2-x+2,$$
$$f^{(2)}(x)=x^4-x^2+2x+1$$

and

$$f^{(3)}(x)=x^5+2x^2+x-1.$$

Then we can write the following Hurwitz matrices for the polynomials $f^{(1)}(x)$, $f^{(2)}(x)$ and $f^{(3)}(x)$ respectively:

$$J^{(1)}=\begin{bmatrix}1&2&0\\1&-1&0\\0&1&2\end{bmatrix},$$

$$J^{(2)}=\begin{bmatrix}0&2&0&0\\1&-1&1&0\\0&0&2&0\\0&1&-1&1\end{bmatrix}$$

and

$$J^{(3)}=\begin{bmatrix}0&2&-1&0&0\\1&0&1&0&0\\0&0&2&-1&0\\0&1&0&1&0\\0&0&0&2&-1\end{bmatrix}.$$

Now we define the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind by using $J^{(1)}$, $J^{(2)}$ and $J^{(3)}$ matrices respectively:

$$\begin{aligned}&x^{(1)}(n)=2x^{(1)}(n-1)-x^{(1)}(n-2)+x^{(1)}(n-3)\ \text{for } n>3\\&\quad\text{where } x^{(1)}(1)=x^{(1)}(2)=0 \text{ and } x^{(1)}(3)=1,\end{aligned}\tag{1}$$

$$\begin{aligned}&x^{(2)}(n)=x^{(2)}(n-2)+2x^{(2)}(n-3)-x^{(2)}(n-4)\ \text{for } n>4\\&\quad\text{where } x^{(2)}(1)=x^{(2)}(2)=x^{(2)}(3)=0 \text{ and } x^{(2)}(4)=1\end{aligned}\tag{2}$$

and

$$\begin{aligned}&x^{(3)}(n)=-x^{(3)}(n-3)+x^{(3)}(n-4)+2x^{(3)}(n-5)\ \text{for } n>5\\&\text{where } x^{(3)}(1)=x^{(3)}(2)=x^{(3)}(3)=x^{(3)}(4)=0 \text{ and } x^{(3)}(5)=1.\end{aligned}\tag{3}$$

By equations (1), (2) and (3), we can write the following companion matrices respectively:

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

$$JH^{(1)} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$JH^{(2)} = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$JH^{(3)} = \begin{bmatrix} 0 & 0 & -1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We call the matrices $JH^{(1)}$, $JH^{(2)}$ and $JH^{(3)}$ by Jacobsthal-circulant-Hurwitz matrices of the first, second and third kind.

Let $x^{(k)}(n)$ be denoted by $x_n^{(k)}$ for $k = 1,2,3$. By inductive argument, we may write for $n \geq 2$:

$$\left(JH^{(1)}\right)^n = \begin{bmatrix} x_{n+3}^{(1)} & x_{n+1}^{(1)} - x_{n+2}^{(1)} & x_{n+2}^{(1)} \\ x_{n+2}^{(1)} & x_n^{(1)} - x_{n+1}^{(1)} & x_{n+1}^{(1)} \\ x_{n+1}^{(1)} & x_{n-1}^{(1)} - x_n^{(1)} & x_n^{(1)} \end{bmatrix},$$

$$\left(JH^{(2)}\right)^n = \begin{bmatrix} x_{n+4}^{(2)} & x_{n+5}^{(2)} & -x_{n+2}^{(2)} + 2x_{n+3}^{(2)} & -x_{n+3}^{(2)} \\ x_{n+3}^{(2)} & x_{n+4}^{(2)} & -x_{n+1}^{(2)} + 2x_{n+2}^{(2)} & -x_{n+2}^{(2)} \\ x_{n+2}^{(2)} & x_{n+3}^{(2)} & -x_n^{(2)} + 2x_{n+1}^{(2)} & -x_{n+1}^{(2)} \\ x_{n+1}^{(2)} & x_{n+2}^{(2)} & -x_{n-1}^{(2)} + 2x_n^{(2)} & -x_n^{(2)} \end{bmatrix}$$

and

$$\left(JH^{(3)}\right)^n = \begin{bmatrix} x_{n+5}^{(3)} & x_{n+6}^{(3)} & x_{n+7}^{(3)} & 2x_{n+3}^{(3)} + x_{n+4}^{(3)} & -2x_{n+4}^{(3)} \\ x_{n+4}^{(3)} & x_{n+5}^{(3)} & x_{n+6}^{(3)} & 2x_{n+2}^{(3)} + x_{n+3}^{(3)} & -2x_{n+3}^{(3)} \\ x_{n+3}^{(3)} & x_{n+4}^{(3)} & x_{n+5}^{(3)} & 2x_{n+1}^{(3)} + x_{n+2}^{(3)} & -2x_{n+2}^{(3)} \\ x_{n+2}^{(3)} & x_{n+3}^{(3)} & x_{n+4}^{(3)} & 2x_n^{(3)} + x_{n+1}^{(3)} & -2x_{n+1}^{(3)} \\ x_{n+1}^{(3)} & x_{n+2}^{(3)} & x_{n+3}^{(3)} & 2x_{n-1}^{(3)} + x_n^{(3)} & -2x_n^{(3)} \end{bmatrix},$$

from which it is clear that $\det\left(JH^{(1)}\right)^n = \left(JH^{(2)}\right)^n = 1$ and $\det\left(JH^{(3)}\right)^n = 2^n$.

It is clear that each of the eigenvalues of the matrices $JH^{(1)}$, $JH^{(2)}$ and $JH^{(3)}$ is distinct. Let $\left\{\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}\right\}$, $\left\{\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}\right\}$ and $\left\{\alpha_1^{(3)}, \alpha_2^{(3)}, \alpha_3^{(3)}, \alpha_4^{(3)}, \alpha_5^{(3)}\right\}$ be the sets of the eigenvalues of the matrices $JH^{(1)}$, $JH^{(2)}$ and $JH^{(3)}$ respectively and let V^k be $(k+2)\times(k+2)$ Vandermonde matrix as follows:

$$V^{(k)} = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{k+1} & \left(\alpha_2^{(k)}\right)^{k+1} & \cdots & \left(\alpha_{k+2}^{(k)}\right)^{k+1} \\ \left(\alpha_1^{(k)}\right)^{k} & \left(\alpha_2^{(k)}\right)^{k} & \cdots & \left(\varepsilon_{k+2}^{(k)}\right)^{k} \\ \vdots & \vdots & & \vdots \\ \alpha_1^{(k)} & \alpha_2^{(k)} & & \alpha_{k+2}^{(k)} \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $k = 1,2,3$. Suppose now that

$$W_i^{(k)} = \begin{bmatrix} \left(\alpha_1^{(k)}\right)^{n+k+2-i} \\ \left(\alpha_2^{(k)}\right)^{n+k+2-i} \\ \vdots \\ \left(\alpha_{k+2}^{(k)}\right)^{n+k+2-i} \end{bmatrix}$$

and $V_{i,j}^{(k)}$ is a $(k+2)\times(k+2)$ matrix obtained from $V^{(k)}$ by replacing the j^{th} column of $V^{(k)}$ by $W_i^{(k)}$. This yields the Binet-type formulas for the Jacobsthal-circulant-Hurwitz matrices of the first, second and third kind, as stated in the following theorem.

Theorem 1. Let $x_n^{(k)}$ be the n^{th} term of the sequence of the k^{th} kind for $k = 1,2,3$. Then

$$h_{ij}^{(k,n)} = \frac{\det V_{i,j}^{(k)}}{\det V^{(k)}},$$

where $\left(JH^{(k)}\right)^n = \left[h_{ij}^{(k,n)}\right]$ such that $k = 1,2,3$.

Proof. Since the eigenvalues of the matrix $JH^{(k)}$ are distinct, it is diagonalisable. Let

$$D^{(1)} = \mathrm{diag}\left(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}\right),$$

$$D^{(2)} = \mathrm{diag}\left(\alpha_1^{(2)}, \alpha_2^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}\right)$$

and

$$D^{(3)} = \mathrm{diag}\left(\alpha_1^{(3)}, \alpha_2^{(3)}, \alpha_3^{(3)}, \alpha_4^{(3)}, \alpha_5^{(3)}\right).$$

Then it is readily seen that $JH^{(k)}V^{(k)} = V^{(k)}D^{(k)}$. Since the matrix $V^{(k)}$ is invertible,

$$\left(V^{(k)}\right)^{-1} JH^{(k)}V^{(k)} = D^{(k)}.$$

Thus, the matrix $JH^{(k)}$ is similar to $D^{(k)}$. So we get

$$\left(JH^{(k)}\right)^n V^{(k)} = V^{(k)}\left(D^{(k)}\right)^n$$

for $n \geq 1$. Then we can write the following linear system of equations for $n \geq 1$:

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

$$\begin{cases} h_{i1}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k+1} + h_{i2}^{(k,n)}\left(\alpha_1^{(k)}\right)^{k} + \cdots + h_{ik+2}^{(k,n)} = \left(\alpha_1^{(k)}\right)^{n+k+2-i} \\ h_{i1}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k+1} + h_{i2}^{(k,n)}\left(\alpha_2^{(k)}\right)^{k} + \cdots + h_{ik+2}^{(k,n)} = \left(\alpha_2^{(k)}\right)^{n+k+2-i} \\ \vdots \\ h_{i1}^{(k,n)}\left(\alpha_{k+2}^{(k)}\right)^{k+1} + h_{i2}^{(k,n)}\left(\alpha_{k+2}^{(k)}\right)^{k} + \cdots + h_{ik+2}^{(k,n)} = \left(\alpha_{k+2}^{(k)}\right)^{n+k+2-i} . \end{cases}$$

So, the following can be obtained:

$$h_{ij}^{(k,n)} = \frac{\det V_{i,j}^{(k)}}{\det V^{(k)}} \text{ for } k = 1,2,3 \text{ and } i, j = 1,2,\ldots,k+2. \qquad \square$$

Then we can give the Binet formulas for the Jacobsthal-circulant-Hurwitz numbers of the first, second and third kind by the following corollary.

Corollary 1. Let $x_n^{(k)}$ be the n^{th} term of the Jacobsthal-circulant-Hurwitz numbers of the first, second and third kind. Then

$$x_n^{(1)} = \frac{\det V_{3,3}^{(1)}}{\det V^{(1)}},$$

$$x_n^{(2)} = -\frac{\det V_{4,4}^{(2)}}{\det V^{(2)}}$$

and

$$x_n^{(3)} = -\frac{\det V_{5,5}^{(3)}}{2\det V^{(3)}}.$$

The generating functions of the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind are, respectively:

$$g^{(1)}(x) = \frac{x^3}{-x^3 + x^2 - 2x + 1},$$

$$g^{(2)}(x) = \frac{x^4}{x^4 - 2x^3 - x^2 + 1}$$

and

$$g^{(3)}(x) = \frac{x^5}{-2x^5 - x^4 + x^3 + 1}.$$

Now we consider the permanent representations of the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind.

Definition 1. A $u \times v$ real matrix $M = \left[m_{i,j}\right]$ is called a contractible matrix in the k^{th} column (resp. row.) if the k^{th} column (resp. row.) contains exactly two non-zero entries.

Let $x_1, x_2, \ldots, x_u$ be row vectors of the matrix M and let M be contractible in the k^{th} column with $m_{i,k} \neq 0$, $m_{j,k} \neq 0$ and $i \neq j$. Then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

Brualdi and Gibson [29] showed that $\operatorname{per}(M)=\operatorname{per}(N)$ if M is a real matrix of order $\alpha>1$ and N is a contraction of M.

Let $P^{(1)}(m)=\left[p_{i,j}^{(1)}\right]$, $P^{(2)}(m)=\left[p_{i,j}^{(2)}\right]$ and $P^{(3)}(m)=\left[p_{i,j}^{(3)}\right]$ be the $m\times m$ super-diagonal matrices defined respectively by

$$p_{i,j}^{(1)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1\le\tau\le m, \\ 1 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-2 \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1,\end{array} \\ -1 & \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le m-1, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge 3$,

$$p_{i,j}^{(2)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-2, \\ 1 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le m-1 \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1,\end{array} \\ -1 & \text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1\le\tau\le m-3, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge 4$, and

$$p_{i,j}^{(3)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau+4 \text{ for } 1\le\tau\le m-4, \\ 1 & \begin{array}{c}\text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1\le\tau\le m-3 \\ \text{and} \\ i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1,\end{array} \\ -1 & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-2, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge 5$. Then we have the following theorem.

Theorem 2. For $k=1,2,3$ and $m\ge k+2$,

$$\operatorname{per}\left(P^{(k)}(m)\right)=x^{(k)}(m+k+2).$$

Proof. Let us consider $k=1$ and let the equation holds for $m\ge 3$. Then we show that the equation holds for $m+1$. If we expand $\operatorname{per}\left(P^{(1)}(m)\right)$ by Laplace expansion of the permanent with respect to the first row, then we obtain

$$\operatorname{per}\left(P^{(1)}(m+1)\right)=2\operatorname{per}\left(P^{(1)}(m)\right)-\operatorname{per}\left(P^{(1)}(m-1)\right)+\operatorname{per}\left(P^{(1)}(m-2)\right).$$

Since $\operatorname{per}\left(P^{(1)}(m)\right)=x^{(1)}(m+3)$, $\operatorname{per}\left(P^{(1)}(m-1)\right)=x^{(1)}(m+2)$ and $\operatorname{per}\left(P^{(1)}(m-2)\right)=x^{(1)}(m+1)$, we easily obtain $\operatorname{per}\left(P^{(1)}(m+1)\right)=x^{(1)}(m+4)$. So the proof is complete.

The proofs for $k=2,3$ are similar to the above and are omitted. □

Maejo Int. J. Sci. Technol. 2020, *14*(01), 56-67

Let $Q^{(1)}(m)=\left[q_{i,j}^{(1)}\right]$, $Q^{(2)}(m)=\left[q_{i,j}^{(2)}\right]$ and $Q^{(3)}(m)=\left[q_{i,j}^{(3)}\right]$ be the $m\times m$ matrices defined respectively by

$$q_{i,j}^{(1)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau \text{ for } 1\le\tau\le m-1, \\ & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-2 \\ 1 & \text{and} \\ & i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1, \\ -1 & \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le m-1, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge3$,

$$q_{i,j}^{(2)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-3, \\ & \text{if } i=\tau \text{ and } j=\tau+1 \text{ for } 1\le\tau\le m-2 \\ 1 & \text{and} \\ & i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1, \\ -1 & \text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1\le\tau\le m-3, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge4$, and

$$q_{i,j}^{(3)}=\begin{cases} 2 & \text{if } i=\tau \text{ and } j=\tau+4 \text{ for } 1\le\tau\le m-4, \\ & \text{if } i=\tau \text{ and } j=\tau+3 \text{ for } 1\le\tau\le m-4 \\ 1 & \text{and} \\ & i=\tau+1 \text{ and } j=\tau \text{ for } 1\le\tau\le m-1, \\ -1 & \text{if } i=\tau \text{ and } j=\tau+2 \text{ for } 1\le\tau\le m-3, \\ 0 & \text{otherwise} \end{cases}$$

where $m\ge5$.

Theorem 3. i. For $m\ge3$,

$$\operatorname{per}\left(Q^{(1)}(m)\right)=x^{(1)}(m).$$

ii. For $m\ge4$,

$$\operatorname{per}\left(Q^{(2)}(m)\right)=-x^{(2)}(m).$$

iii. For $m\ge5$,

$$\operatorname{per}\left(Q^{(3)}(m)\right)=2x^{(3)}(m).$$

Proof. Let us consider the matrix $Q^{(2)}(m)$ and let the equation hold for $m\ge4$. Then we show that the equation holds for $m+1$. If we expand $\operatorname{per}\left(Q^{(2)}(m)\right)$ by Laplace expansion of the permanent according to the first row, then we obtain

$$\operatorname{per}\left(Q^{(2)}(m+1)\right)=\operatorname{per}\left(Q^{(2)}(m-1)\right)+2\operatorname{per}\left(Q^{(2)}(m-2)\right)-\operatorname{per}\left(Q^{(2)}(m-3)\right).$$

Also, since $\text{per}\left(Q^{(2)}(m-1)\right)=-x^{(2)}(m-1)$, $\text{per}\left(Q^{(2)}(m-2)\right)=-x^{(2)}(m-2)$ and $\text{per}\left(Q^{(2)}(m-3)\right)=-x^{(2)}(m-3)$, it is clear that $\text{per}\left(Q^{(2)}(m+1)\right)=-x^{(2)}(m+1)$.

The proofs for the matrices $Q^{(1)}(m)$ and $Q^{(3)}(m)$ are similar. □

Assume that the $m\times m$ matrices $R^{(k)}(m)=\left[r_{i,j}^{(k)}\right]$ for $k=1,2,3$ are defined by

$$R^{(k)}(m)=\begin{bmatrix} 1 & \cdots & \underset{\downarrow}{\overset{(m-k-2)^{\text{th}}}{1}} & 0 & & 0 \\ 1 & & & & & \\ 0 & & & & & \\ 0 & & & Q^{(k)}(m-1) & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \quad \text{for } m>k+2.$$

Then we can give more general results by using other permanent representations than the above.

Theorem 4. i. For $m>3$,

$$\text{per}\left(R^{(1)}(n)\right)=\sum_{i=1}^{m-1}x^{(1)}(i).$$

ii. For $m>4$,

$$\text{per}\left(R^{(2)}(n)\right)=-\sum_{i=1}^{m-1}x^{(2)}(i).$$

iii. For $m>5$,

$$\text{per}\left(R^{(3)}(n)\right)=2\sum_{i=1}^{m-1}x^{(3)}(i).$$

Proof. i. If we extend $\text{per}\left(R^{(1)}(m)\right)$ with respect to the first row, we can write

$$\text{per}\left(R^{(1)}(m)\right)=\text{per}\left(R^{(1)}(m-1)\right)+\text{per}\left(Q^{(1)}(m-1)\right).$$

Thus, by the results and an inductive argument, the proof is easily seen.

The proofs for the matrices $R^{(2)}(m)$ and $R^{(3)}(m)$ are similar. □

Let the notation $A\circ K$ denote the Hadamard product of A and K. A matrix A is called convertible if there is an $m\times m$ (1, -1)-matrix K such that $\text{per}(A)=\det(A\circ K)$.

Let $m>k+2$ and let T be the $m\times m$ matrix defined by

$$T=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

It is easy to see that $\operatorname{per}\left(P^{(k)}(m)\right)=\det\left(P^{(k)}(m)\circ T\right)$, $\operatorname{per}\left(Q^{(k)}(m)\right)=\det\left(Q^{(k)}(m)\circ T\right)$ and $\operatorname{per}\left(R^{(k)}(m)\right)=\det\left(R^{(k)}(m)\circ T\right)$. Then we have the following useful results.

Corollary 2. i. $\det\left(P^{(k)}(m)\circ T\right)=x^{(k)}(m+k+2)$, for $k=1,2,3$.

ii. $\det\left(Q^{(1)}(m)\circ T\right)=x^{(1)}(m)$ for $m>3$,

$\det\left(Q^{(2)}(m)\circ T\right)=-x^{(2)}(m)$ for $m>4$

and $\det\left(Q^{(3)}(m)\circ T\right)=2x^{(3)}(m)$ for $m>5$.

iii. $\det\left(R^{(1)}(m)\circ T\right)=\sum_{i=1}^{m-1}x^{(1)}(i)$ for $m>3$,

$\det\left(R^{(2)}(m)\circ T\right)=-\sum_{i=1}^{m-1}x^{(2)}(i)$ for $m>4$

and $\det\left(R^{(3)}(m)\circ T\right)=2\sum_{i=1}^{m-1}x^{(3)}(i)$ for $m>5$.

Now we consider the sums of Jacobsthal-circulant-Hurwitz numbers of the first, second and third kind. Let

$$S_n=\sum_{i=1}^{n}x^{(k)}(i)$$

for $i\geq 1$ and $k=1,2,3$. Suppose that $\left(T^{(k)}\right)^n$ are the $(k+2)\times(k+2)$ matrices such that

$$T^{(k)}=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & JH^{(k)} & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then it can be shown by induction that

$$\left(T^{(k)}\right)^{n}=\begin{bmatrix} 1 & 0 \;\; 0 & \cdots & 0 \;\; 0 \\ S_{n+k+1} & & & \\ S_{n+k} & & & \\ \vdots & & \left(JH^{(k)}\right)^{n} & \\ S_{n+1} & & & \\ S_{n} & & & \end{bmatrix}.$$

CONCLUSIONS

We have defined the Jacobsthal-circulant-Hurwitz sequences of the first, second and third kind. Using the roots of their characteristic polynomials, we have produced their Binet formulas. Furthermore, we have given the generating functions, the permanent and determinant representations and the sums of these sequences.

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