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Communication

Invex and quasi-invex NCP functions and their properties

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Abstract: A class of invex nonlinear complementarity (NCP) functions is introduced. The existence of such a function is illustrated by citing nontrivial examples. Some interesting properties of invex NCP functions have been developed. Quasi-invex NCP functions and their properties are also presented.

Keywords: nonlinear complementarity function, stationary point, invex functions

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INTRODUCTION

Mangasarian [1] was the first to introduce the idea of nonlinear complementarity (NCP) functions, which are useful in many unconstrained and constrained optimisation problems. Later on, several researchers enhanced this idea and proved some more properties related to NCP functions. Miri and Effati [2] studied the quasi-convexity of NCP functions and some properties of homogeneous NCP functions and proved that there does not exist any pseudo-convex NCP function. Thus, differentiability and convexity cannot hold simultaneously for NCP functions. Later on, Huang et al. [3] established that such result also holds for generalised complementarity problems. Galantai [4] developed various methods for constructing new NCP functions. Abdullah et al. [5] constructed a model related to the absolute-value equation and further solved it using complementarity and smoothing techniques. Recently, Ranjbar et al. [6] presented a one-layer neural network model for solving the convex optimisation problems using Karush–Kuhn–Tucker conditions, and Mangasarian and Solodov [7] discussed implicit Lagrangian as NCP function. A family of neural networks for solving NCP was considered by Alcantara and Chen [8] who also analysed the stability of the proposed scheme and further provided numerical simulations.

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Hanson [9] introduced the concept of invexity replacing the difference vector $(x - \bar{x})$ in the definition of a convex function by any vector function η and established Karush-Kuhn-Tucker type sufficient optimality condition for a nonlinear optimisation problem. Craven and Glover [10] proved that the set of functions whose global minima are attained only on the stationary points is equivalent to the set of invex functions. Kaul and Kaur [11] defined quasi-convex, convex and pseudo-convex functions and some sufficient optimality conditions are proved for a nonlinear programming problem.

This paper presents some properties related to partial derivatives of NCP functions. It also focuses on the invex NCP functions, their properties and related results. By citing a nontrivial example, the existence of such functions has also been shown. An important property that "*Every invex NCP function is not a convex function*" has been proved. Some more properties of invex NCP and quasi-invex NCP functions are also discussed.

PRELIMINARIES

In this section we recall some definitions and theorems to be used.

Definition 1 [4]**.** A function $\emptyset: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is called an NCP function if and only if

$$\emptyset(\alpha, \beta) = 0 \Leftrightarrow \alpha \geq 0, \alpha\beta = 0.$$

Some of the examples of NCP functions are

$$\emptyset_{min}(\alpha, \beta) = \min\{\alpha, \beta\} \text{ [12]},$$

$$\emptyset_{FB} = \sqrt{\alpha^2 + \beta^2} - (\alpha + \beta) \text{ [13]},$$

$$\emptyset_W = (\alpha - \beta)^+ - \alpha \text{ [14]},$$

$$\phi_{MS} = \alpha\beta + \frac{1}{2\lambda}\{[(\alpha - \lambda\beta)^+]^2 - \alpha^2 + [(\beta - \lambda\alpha)^+]^2 - \beta^2\} \, (\lambda > 1) \text{ [7]}$$

where $x^+ = \max\{x, 0\}$ and $x^- = \min\{x, 0\}$.

Definition 2. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function on $\mathbb{R}^n$. Then the function f is said to be pseudo-convex if and only if $\forall\, a, b \in \mathbb{R}^n$ with $f(a) < f(b)$; we have $\nabla f(b)^T(a - b) < 0$.

Miri and Effati [2] studied some properties of generalised convex NCP functions. Below, we state a lemma and a theorem, which are needed in the paper. For proof and more details, we refer to Miri and Effati [2].

Lemma 1. Suppose that f is an NCP function. If the first-order partial derivatives of f exist at (0,0), then $\nabla f(0,0) = (0,0)^T$.

Theorem 1. There is no pseudo-convex NCP function.

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PARTIAL DERIVATIVES OF NCP FUNCTIONS

Definition 3. A function $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be homogeneous of degree $\beta \in \mathbb{R}$ if and only if

$$h(ka) = k^\beta h(a)$$

for each $a \in \mathbb{R}^n$ and $k > 0$.

Lemma 2 [2]. Let ϕ be a NCP homogeneous function of degree α. Then the first-order partial derivatives of ϕ exist at the origin if and only if $\alpha > 1$.

Remark 1. Following lines of Lemma 2, we can state the following result:

Let $\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a homogeneous function of degree α such that ϕ_x and ϕ_y exist for all $(x, y) \in \mathbb{R}^2$. If ϕ_x and ϕ_y are NCP functions, then ϕ_{xx} and ϕ_{yy} attain zero values at the origin if and only if $\alpha > 2$.

Let a set Z be given by

$$Z := \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha \geq 0, \beta \geq 0, \alpha\beta = 0\}.$$

Clearly, if f is an NCP function, then $f(\alpha, \beta) = 0, \forall(\alpha, \beta) \in Z$. Further, suppose that

$$Z_1 = \{(\alpha, 0) \in \mathbb{R}^2 | \alpha \geq 0\} \text{ and}$$

$$Z_2 = \{(0, \beta) \in \mathbb{R}^2 | \beta \geq 0\} \subset Z.$$

Then obviously $Z = Z_1 \cup Z_2$ and $Z_1 \cap Z_2 = \{(0,0)\}$.

Theorem 2. Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an NCP function and the first-order partial derivatives of f exist on Z. Then

$$\left(\frac{\partial \mathrm{f}}{\partial \mathrm{x}}\right)_{\mathrm{Z}_1} = \left(\frac{\partial \mathrm{f}}{\partial \mathrm{y}}\right)_{\mathrm{Z}_2} = 0.$$

Proof. Let $(\alpha, 0) \in Z_1$. Then

$$\frac{\partial f}{\partial x}(\alpha, 0) = \lim_{h \to 0^+} \frac{f(a+h, 0) - f(a, 0)}{h} = 0.$$

(Since f is an NCP function, therefore $f(a, 0) = 0 \ \forall \ (a, 0) \in Z_1$). Similarly, for $(0, b) \in Z_2$, we obtain

$$\frac{\partial f}{\partial y}(0, b) = 0.$$

Hence the proof.

Remark 2. It may be noted that if f is an NCP function and the first-order partial derivatives of f exist on Z, then $\left(\frac{\partial f}{\partial x}\right)$ on $Z_2 \setminus \{(0,0)\}$ may not be equal to zero. For example

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}, f(a, b) = \sqrt{a^2 + b^2} - (a + b)$$

is an NCP function [13], but $\left(\frac{\partial f}{\partial x}\right)$ on $Z_2 \setminus \{(0,0)\} = -1$. A similar argument for $\left(\frac{\partial f}{\partial y}\right)$ on $Z_1 \setminus \{(0,0)\}$ also holds true.

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Let F^* be the set of all $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that f is an NCP function and $\left(\frac{\partial f}{\partial x}\right)_Z = \left(\frac{\partial f}{\partial y}\right)_Z = 0$. Clearly, the set $F^* \neq \emptyset$ since $\emptyset_{MS} \in F^*$ [15].

INVEX NCP FUNCTIONS AND THEIR PROPERTIES

The class of invex functions is a generalised class of differentiable convex functions. In this section we first define invex NCP function and illustrate it by examples.

Definition 4 [16]**.** Let $X \subset \mathbb{R}^n$ be a non-empty set. A differentiable function $f: X \longrightarrow \mathbb{R}$ is said to be invex (η convex) at $b \in X$ if there exists a function $\eta: X \times X \rightarrow \mathbb{R}^n$ such that

$$f(a) - f(b) \geq \eta^T(a,b)\nabla f(b), \forall a \in X.$$

If the above definition holds for all $b \in X$, then the function f is called invex on X. The properties of an invex function [16] are reflected in the following theorem and corollary. For their proofs and other details, we refer to Israel and Mond [16].

Theorem 3. f is an invex function if and only if every stationary point is global minimum.

Corollary 1. If f has no stationary point, then f is an invex function.

The following are examples of invex NCP functions.

Example 1. Let $f_{YFB}: \mathbb{R}^2 \backslash \{(0,0)\} \longrightarrow \mathbb{R}$ be defined as:

$$f_{YFB}(\alpha,\beta) = (\sqrt{\alpha^2+\beta^2} - (\alpha+\beta))^2.$$

The function f_{YFB} is an NCP function [17]. Next, to prove that f_{YFB} is an invex function, we shall use Theorem 3. Now,

$$\frac{\partial f_{YFB}}{\partial \alpha} = 2(\sqrt{\alpha^2\beta^2} - (\alpha+\beta))\left(\frac{\alpha - \sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2}}\right).$$

Similarly,

$$\frac{\partial f_{YFB}}{\partial \beta} = 2(\sqrt{\alpha^2+\beta^2} - (\alpha+\beta))\left(\frac{\beta - \sqrt{\alpha^2+\beta^2}}{\sqrt{\alpha^2+\beta^2}}\right).$$

Then the stationary points of function f_{YFB} are given by

$$\frac{\partial f_{YFB}}{\partial \alpha} = \frac{\partial f_{YFB}}{\partial \beta} = 0.$$

This yields $\alpha \geq 0, \beta \geq 0$ and $\alpha\beta = 0$. Since f_{YFB} is an NCP function, therefore, the value of function f_{YFB} at $\alpha, \beta \geq 0$ and $\alpha\beta = 0$ is zero. Further, since $f_{YFB} \geq 0, \forall(\alpha,\beta) \in \mathbb{R}^2 \backslash \{(0,0)\}$, therefore, stationary points are global minima of the function. Hence f_{YFB} is an invex NCP function.

Example 2.

$$\phi_{MS} = \alpha\beta + \frac{1}{2\lambda}\{[(\alpha-\lambda\beta)^+]^2 - \alpha^2 + [(\beta-\lambda\alpha)^+]^2 - \beta^2\}\ (\lambda > 1)$$

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where $\xi^+ = \max\{\xi, 0\}$.

Example 3.

$$f_{SW}(\alpha,\beta) = \begin{cases} (\sqrt{\alpha^2+\beta^2} - (\alpha+\beta))^2, & \alpha \geq 0, \beta \geq 0 \\ (\alpha^-)^2 + (\beta^-)^2, & \text{otherwise} \end{cases}$$

where $\xi^- = \min\{\xi, 0\}$.

Next we develop some properties of invex NCP functions. We now state Theorems 4 and 5 related with the properties of invex NCP functions whose proofs can be obtained easily.

Theorem 4. For any constant $c > 0$, f is invex NCP $\Leftrightarrow$ cf is invex NCP.

Theorem 5. Let f_1 and f_2 be two invex NCP functions with respect to the same η. Then $f_1 + f_2$ is also invex NCP function if $sgn(f_1) = sgn(f_2)$.

Theorem 6. Every invex NCP function is not a convex function.

Proof. On the contrary, suppose there exists an NCP invex function f which is convex also. Since every invex function is differentiable and we know that every differentiable convex function is pseudo-convex, therefore f is a pseudo-convex NCP function. However, by Theorem 1, there does not exist any pseudo-convex NCP function. This is a contradiction as an NCP function is invex and convex simultaneously. Hence the proof.

Theorem 7. Let $f \in F^*$ be a non-negative differentiable function. Then f is an invex NCP function for all $(\alpha,\beta) \in Z$.

Proof. Let $(\alpha,\beta) \in Z$. Then $f(\alpha,\beta) = 0$ and $\nabla f(\alpha,\beta) = (0,0)^T$ as $f \in F^*$. Therefore

$$\eta^T[(x,y),(\alpha,\beta)]\nabla f(\alpha,\beta) = 0. \quad (1)$$

As f is a non-negative function for all $(x,y) \in \mathbb{R}^2$, therefore

$$f(x,y) \geq 0 = f(\alpha,\beta).$$

This implies

$$f(x,y) - f(\alpha,\beta) \geq 0,$$

which together with (1) yields

$$f(x,y) - f(\alpha,\beta) \geq \eta^T[(x,y),(\alpha,\beta)]\nabla f(\alpha,\beta).$$

Hence f is an invex NCP function on Z.

Theorem 8. Let f be a non-negative NCP homogeneous differentiable function of degree $\alpha > 1$. Then f is an invex function at (0,0).

Proof. Let f be a non-negative NCP homogeneous differentiable function of degree $\alpha > 1$. Then from Lemma 2, partial derivatives of f exist at the origin. Also, using Lemma 1, $\nabla f(0,0) = (0,0)^T$. Therefore

$$\eta^T[(x,y),(0,0)]\nabla f(0,0) = 0. \quad (2)$$

Also, $f(x,y) \geq 0 = f(0,0)$. Further, it follows from (2) that

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$$f(x,y) - f(0,0) \geq \eta^T[(x,y),(0,0)]\nabla f(0,0).$$

Hence f is an invex function at (0,0).

Theorem 9. Let $f: \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ be an NCP function. Suppose that first-order partial derivatives of function f exist for all $(x,y) \in \mathbb{R}^2$. If f_x and f_y are NCP functions, then f is an invex function.

Proof. Let $(a,b) \in \mathbb{R}^2$. Since f_x and f_y are NCP functions, therefore

$$f_x(a,b) = 0 \Rightarrow a \geq 0, b \geq 0, ab = 0 \Rightarrow f_y(a,b) = 0.$$

So the points $a, b \geq 0$ and $ab = 0$ are stationary points of function f. Moreover, since f is an NCP function, therefore f attains a minimum value at these points. Hence every stationary point is a global minimum. Therefore f is an invex function.

QUASI-INVEX NCP FUNCTIONS AND THEIR PROPERTIES

Definition 5 [10]**.** A differentiable function $g: X \to \mathbb{R}$ $(X \subset \mathbb{R}^n)$ is said to be quasi-invex (η quasi-convex) at $y \in X$ if there exists a function $\eta: X \times X \to \mathbb{R}^n$ such that

$$f(x) \leq f(y) \Rightarrow \eta^T(x,y)\nabla f(y) \leq 0, \forall x \in X.$$

If the above definition holds for all $x, y \in X$, then function f is called quasi-invex on X.

Theorem 10. Let $f \in F^*$ be a differentiable function. Then f is a quasi-invex NCP function for all $(\alpha,\beta) \in Z$.

Proof. Let $(\alpha,\beta) \in Z$. Then $f(\alpha,\beta) = 0$ and $\nabla f(\alpha,\beta) = (0,0)^T$ as $f \in F^*$. Since $\nabla f(\alpha,\beta) = (0,0)^T$, therefore

$$\eta^T[(x,y),(\alpha,\beta)]\nabla f(\alpha,\beta) = 0.$$

Thus, $f(x,y) \leq f(\alpha,\beta)$ always implies that $\eta^T((x,y),(\alpha,\beta))\nabla f(\alpha,\beta) = 0$ for all $x, y \in \mathbb{R}$. Hence f is quasi-invex at (α,β). Then f is a quasi-invex NCP function for all $(\alpha,\beta) \in Z$.

Remark 3. It is to be noted (from Theorems 7 and 10) that if $f \in F^*$ is a negative differentiable function, then f is a quasi-invex NCP function, not an invex NCP function.

Theorem 11. Let f be an NCP homogeneous differentiable function of degree $\alpha > 1$. Then f is a quasi-invex function at (0,0).

Proof. From Lemmas 1 and 2, $\nabla f(0,0) = (0,0)^T$. Hence

$$\eta^T[(x,y),(0,0)]\nabla f(0,0) = 0.$$

Therefore $f(x,y) \leq f(0,0)$ yields

$$\eta^T((x,y),(0,0))\nabla f(0,0) = 0$$

for all $x, y \in \mathbb{R}$. Hence f is a quasi-invex function at (0,0).

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