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Generalisation of close-to-convex functions associated with Janowski functions

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Abstract: A new class of q-close-to-convex functions associated with Janowski functions is defined. In this regard, we give sufficient conditions and prove the famous de Branges theorem for this newly-defined class of q-close-to-convex functions. We also give the application of our results to finding sufficient conditions for the celebrated Mittag-Leffler function to be a Janowski q-close-to-convex function.

Keywords: univalent functions, close-to-convex functions, q-derivative operator, q-close-to-convex function, Bieberbach conjecture, de-Branges theorem

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INTRODUCTION AND PRELIMINARIES

By $\mathrm{H}(\mathrm{U})$ we denote the class of functions which are analytic in the open unit disk

$$\mathrm{U}=\{z : z \in \mathrm{C} \quad \text{and} \quad |z|<1\},$$

where C is, as usual, the complex plane. Let A denote the class of functions having the following form:

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$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (\forall\, z \in \mathrm{U}), \tag{1}$$

which are in the open unit disk U, centred at the origin and normalised by the conditions given by

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Also, let $\mathrm{S} \subset \mathrm{A}$ be the class of functions which are univalent in U.

Furthermore, we denote by S^*, the class of functions in A which are starlike in U and satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad (\forall z \in \mathrm{U}).$$

For $f \in \mathrm{S}^*$, one can find that [1]

$$|a|_n \leq n \quad for\ n = 2,3,\dots. \tag{2}$$

Moreover, the class of close-to-convex functions in U are denoted here by K and defined as follows. A function $f \in \mathrm{A}$ is said to be in the class K if and only if there exists a function $g \in S^*$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \quad (\forall z \in \mathrm{U}). \tag{3}$$

Furthermore, if two functions f and g are analytic in U, we say that the function f is subordinate to g and written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function w which is analytic in U with

$$w(0) = 0 \ and\ |w(z)| < 1$$

such that

$$f(z) = g(w(z)).$$

It can also be seen that if the function g is univalent in U, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathrm{U}) \Rightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

We next denote by P the class of analytic functions p which are normalised by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{4}$$

such that

$$\Re(p(z)) > 0.$$

Definition 1. A given analytic function h with $h(0) = 1$ is said to belong to the class $P[A,B]$ if and only if

$$h(z) \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1.$$

The analytic function class $\mathrm{P}[A,B]$ was introduced by Janowski [2], who showed that $h(z) \in P[A,B]$ if and only if there exists a function $p \in \mathrm{P}$ such that

$$h(z)=\frac{(A+1)p(z)-(A-1)}{(B+1)p(z)-(B-1)},\ -1\le B<A\le 1.$$

Definition 2. A function $f\in A$ is said to belong to the class $K[A,B]$ if and only if there exists $g\in S^*$ such that

$$\frac{zf'(z)}{g(z)}=\frac{(A+1)p(z)-(A-1)}{(B+1)p(z)-(B-1)},\ -1\le B<A\le 1. \tag{5}$$

We now recall some basic definitions and concept details of the q-calculus which are used in this article. We suppose throughout the article that $0<q<1$ and that

$$N=\{1,2,3...\}=N_0\setminus\{0\} \qquad (N_0:=\{0,1,2,3...\}).$$

Definition 3. Let $q\in(0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q=\begin{cases}\frac{1-q^\lambda}{1-q} & (\lambda\in C)\\ \sum_{k=1}^{n-1}q^k=1+q+q^2+\cdots+q^{n-1} & (\lambda=n\in C).\end{cases}$$

Definition 4. Let $q\in(0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q!=\begin{cases}1 & (n=0)\\ \prod_{k=1}^{n}[k]_q & (n\in N).\end{cases}$$

Definition 5 [3, 4]**.** The q-derivative (or q-difference) D_q of a function f is defined in a given subset of C by

$$D_qf(z)=\begin{cases}\frac{f(z)-f(qz)}{(1-q)z} & (z\neq 0)\\ f'(0) & (z=0).\end{cases} \tag{6}$$

We note from (6) that the q-derivative (or the q-difference) operator D_qf converges to the ordinary derivative operator as follows:

$$\lim_{q\to 1-}(D_qf)(z)=\lim_{q\to 1-}\frac{f(z)-f(qz)}{(1-q)z}=f'(z).$$

For a differentiable function f in a given subset of C, it is readily deduced from (1) and (6) that

$$D_qf(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1} \tag{7}$$

In Geometric Function Theory, several subclasses belonging to the class A of normalised analytic functions A have already been investigated in different aspects. The above defined q-calculus gives valuable tools that have been extensively used for investigating several subclasses of A. Ismail et al. [5] were the first who employed q-derivative operator D_q to study the q-calculus analogous with the class S^* of starlike functions in U. Raghavendar and Swaminathan [6] used the q-derivative operator D_q for studying the q-calculus corresponding to the class K of close-to-convex functions in U (see Definition 6 below).

Recently, using the q-deravative operator, certain subclasses of analytic and bi-univalent functions were introduced and investigated [7-9]. For exmaple, non-sharp estimates on the first two

Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were studied [10]. Kanas and Raducanu [11] have used the fractional q-calculus operators in the investigation of certain classes of functions which are analytic in open-unit disk U by using the idea of canonical domain. Coefficient inequality for q-closed-to-convex functions with respect to Janowski-type starlike functions has been studied by Ucar [12]. In fact, historically speaking, a remarkably significant usage of the q-calculus in the context of Geometric Function Theory of Complex Analysis was basically furnished and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava [13] and Srivastava and Bansal [14].

Definition 6 [6]**.** A function $f \in \mathrm{A}$ is said to belong to the class K_q if there exists $g \in S^*$ such that

$$f(0) = f'(0) - 1 = 0 \tag{8}$$

and

$$\left| \frac{z}{g(z)} (D_q f)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \quad (\forall z \in \mathrm{U}). \tag{9}$$

Then we say that $f \in \mathrm{K}_q$ with the function g. We note that the notation K_q was first used by Sahoo and Sharma [15]. It is readily observed that, as $q \to 1^-$, the closed disk

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

becomes the right-half plane and the class K_q of q-close-to-convex function reduces to the familiar class K. Equivalently, by using the principle of subordination between analytic functions, we can rewrite the conditions in (8) and (9) as follows [9]:

$$\frac{z}{g(z)} (D_q f)(z) \prec \tilde{p} \qquad \left(\tilde{p} = \frac{1+z}{1-qz} \right).$$

Motivated by the work of Janowski [2], Ucar [12] and other related works cited above in this paper, we shall consider a (presumably new) subclass of q-close-to-covex function with respect to Janowski functions.

Definition 7. A function $f \in \mathrm{A}$ is said to belong to the class $K_q[A, B]$ if and only if there exists $g \in S^*$ such that

$$\frac{D_q f(z)}{g(z)} = \frac{(A+1)\tilde{p} - (A-1)}{(B+1)\tilde{p} - (B-1)}, \quad -1 \leq B < A \leq 1, \ q \in (0,1)$$

which, by using the principle of subordination between analytic functions, can be written as

$$\frac{z D_q f(z)}{g(z)} \prec \phi(z),$$

where

$$\phi(z) = \frac{z(A+1) + 2 + zq(A-1)}{z(B+1) + 2 + zq(B-1)}, \quad -1 \leq B < A \leq 1, q \in (0,1).$$

Or, equivalently, $f \in K_q[A, B]$ if and only if there exists $g \in S^*$ such that

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$$\left|\frac{(B-1)\frac{D_q f(z)}{g(z)}-(A-1)}{(B+1)\frac{D_q f(z)}{g(z)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}.$$

Remark 1. Firstly, if we let $q \to 1$, we have the familiar $K[A,B]$ (see Definition 2) introduced and studied by Noor [16]. Secondly, for $A=1$, $B=-1$, we have K_q introduced and studied by Raghavendar et al. [6]. Thirdly, for $A=1$, $B=-1$ and if we let $q \to 1$, we have K, the class of close-to-convex functions introduced and studied by Kaplan [17].

The following Lemma will be required for the proof of our main results.

Lemma 1 [18]**.** Let the function p given by

$$p(z)=1+\sum_{n=1}^{\infty} p_n z^n$$

be subordinate to H given by

$$H(z)=1+\sum_{n=1}^{\infty} C_n z^n.$$

If $H(z)$ is univalent in U and $\mathrm{H}(\mathrm{U})$ is convex, then

$$|p_n| \le |C_1|, \quad n \in \mathrm{N}.$$

MAIN RESULTS

In this section we prove our main results. Throughout our discussion, we assume that

$$-1 \le B < A \le 1, \quad \text{and} \quad q \in (0,1).$$

Theorem 1. A function $f \in \mathrm{A}$ of the form given by (1) is in the class $\mathrm{K}_q[A,B]$ if it satisfies the following condition:

$$\sum_{n=2}^{\infty}\left|[n]_q (B-1)a_n-(A-1)n\right|<|B-A|. \tag{10}$$

Proof. Assuming that the inequality (10) holds true, it suffices to show that

$$\left|\frac{(B-1)\frac{zD_q f(z)}{g(z)}-(A-1)}{(B+1)\frac{zD_q f(z)}{g(z)}-(A+1)}-\frac{1}{1-q}\right|<\frac{1}{1-q}. \tag{11}$$

Letting

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n \qquad (\forall\, z \in \mathrm{U}), \tag{12}$$

we have

$$\left|\frac{(B-1)\frac{zD_q f(z)}{g(z)}-(A-1)}{(B+1)\frac{zD_q f(z)}{g(z)}-(A+1)}-\frac{1}{1-q}\right| \tag{13}$$

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$$\leq \left|\frac{(B-1)zD_qf(z)-(A-1)g(z)}{(B+1)zD_qf(z)-(A+1)g(z)}-1\right|+\frac{q}{1-q}$$

$$=2\left|\frac{g(z)-zD_qf(z)}{(B+1)zD_qf(z)-(A+1)g(z)}\right|+\frac{q}{1-q}$$

$$=2\left|\frac{\sum_{n=2}^{\infty}\left(b_n-[n]_q a_n\right)z^n}{(B-A)+\sum_{n=2}^{\infty}\left\{[n]_q(B+1)a_n-(A+1)b_n\right\}z^n}\right|+\frac{q}{1-q}.$$

Moreover, by using trigonometric inequality and (2), we have

$$\left|\frac{(B-1)\frac{zD_qf(z)}{g(z)}-(A-1)}{(B+1)\frac{zD_qf(z)}{g(z)}-(A+1)}-\frac{1}{1-q}\right|$$

$$\leq 2\cdot\frac{\sum_{n=2}^{\infty}\left|n-[n]_q a_n\right|}{|B-A|-\sum_{n=2}^{\infty}\left|[n]_q(B+1)a_n-(A+1)n\right|}+\frac{q}{1-q}. \tag{14}$$

The last expression in (14) is bounded above by $\frac{1}{1-q}$ if

$$\sum_{n=2}^{\infty}\left|[n]_q(B-1)a_n-(A-1)n\right|<|B-A|.$$

Thus, we have completed the proof of Theorem 1.

Theorem 2. Let $f\in \mathrm{K}_q[A,B]$ be of the form (1). Then for $n\geq 2$,

$$|a_n|\leq\frac{1}{[n]_q}\left[n+\frac{(A-B)(q+1)}{2}\prod_{j=0}^{n-2}(j+1)\right]. \tag{15}$$

Proof. By definition, for $f\in \mathrm{K}_q[A,B]$, we have

$$\frac{zD_qf(z)}{g(z)}=p(z), \tag{16}$$

where

$$p(z)\prec\frac{z(A+1)+2+zq(A-1)}{z(B+1)+2+zq(B-1)}$$

$$=1+\tfrac{1}{2}(A-B)(q+1)z+\tfrac{1}{4}(A-B)(q+1)\{(q+1)B-q+1\}z^2+\cdots.$$

Since

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n,$$

then by Lemma 1 we have

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$$\left|p_n\right| \le \frac{1}{2}(A-B)(q+1), \; n \ge 1. \tag{17}$$

Now from (16), we have

$$zD_q f(z) = p(z)g(z), \tag{18}$$

which implies that

$$z+\sum_{n=2}^{\infty}[n]_q a_n z^n = \left(z+\sum_{n=2}^{\infty} b_n z^n\right)\left(1+\sum_{n=1}^{\infty} c_n z^n\right).$$

Equating the coefficients of z^n on both sides, we have

$$[n]_q a_n = b_n + \sum_{j=1}^{n-1} a_{n-j} c_j, \qquad a_1 = 1.$$

This implies that

$$\left|a_n\right| \le \frac{1}{[n]_q}\left[\left|b_n\right| + \sum_{j=1}^{n-1}\left|b_{n-j}\right|\left|c_j\right|\right], \quad a_1 = 1.$$

Moreover, by using (17) and (2), we have

$$\left|a_n\right| \le \frac{1}{[n]_q}\left[n + \frac{(A-B)(q+1)}{2}\sum_{j=1}^{n-1} j\right], \qquad a_1 = 1. \tag{19}$$

Next, in order to prove that

$$\frac{1}{[n]_q}\left[n + \frac{(A-B)(q+1)}{2}\sum_{j=1}^{n-1} j\right] \le \frac{1}{[n]_q}\left[n + \frac{(A-B)(q+1)}{2}\prod_{j=0}^{n-2}(j+1)\right], \tag{20}$$

we use the principle of mathematical induction. Of course, for $n = 2$, we find from (19) that

$$\left|a_2\right| \le \frac{1}{[2]_q}\left[2 + \frac{(q+1)(A-B)}{2}\right],$$

which results also from (15). Now for $n = 3$, we find from (19) that

$$\left|a_3\right| \le \frac{1}{[3]_q}\left[3 + \frac{(q+1)(A-B)}{2} + \frac{2(A-B)(1+q)}{2}\right]$$

$$= \frac{1}{[3]_q}\left[3 + \frac{(q+1)(A-B)}{2}(1+2)\right],$$

which follows also from (15). Let the hypothesis be true for $n = m.$ Then it follows from (19) that

$$\left|a_m\right| \le \frac{1}{[m]_q}\left[m + \frac{(A-B)(q+1)}{2}\sum_{j=1}^{m-1} j\right] \qquad a_1 = 1.$$

On the other hand, from (15), we have

$$\left|a_m\right| \le \frac{1}{[m]_q}\left[m + \frac{(A-B)(q+1)}{2}\prod_{j=0}^{m-2}(j+1)\right].$$

By the induction hypothesis, we have

$$\frac{1}{[m]_q}\left[m+\frac{(A-B)(q+1)}{2}\sum_{j=1}^{m-1} j\right] \le \frac{1}{[m]_q}\left[m+\frac{(A-B)(q+1)}{2}\prod_{j=0}^{m-2}(j+1)\right].$$

We now consider

$$|a_{m+1}| \le \frac{1}{[m+1]_q}\left[m+1+\frac{(A-B)(q+1)}{2}\sum_{j=1}^{m} j\right]$$

$$=\frac{1}{[m+1]_q}\left[m+1+\frac{(A-B)(q+1)}{2}(1+2+\ldots+m)\right]$$

$$=\frac{1}{[m+1]_q}\left[m+1+\frac{(A-B)(q+1)}{2}\prod_{j=0}^{m-1}(j+1)\right].$$

Also from (15), we have

$$|a_{m+1}| \le \frac{1}{[m+1]_q}\left[m+1+\frac{(A-B)(q+1)}{2}\prod_{j=0}^{m-1}(j+1)\right],$$

which shows that inequality (20) is true for $n=m+1$. Thus, by the principle of mathematical induction, we have completed the proof of Theorem 2.

DE BRANGES THEOREM FOR $\mathrm{K}_q[A,B]$

In this section Theorem 3 works as one of the key results for estimating coefficient bounds for series representation of functions in the class $\mathrm{K}_q[A,B]$. In other words, we investigate the famous Bieberbach conjecture problem on coefficients of analytic q-close-to-convex functions associated with the Janowski functions. The Bieberbach conjecture for close-to-covex functions is given by Reade [19].

We now continue to give the Bieberbach-deBranges Theorem for functions in the q-close-to-convex family associated with the Janowski functions.

Theorem 3. Let $f \in \mathrm{K}_q[A,B]$ be of the form (1). Then for $n \ge 2$,

$$|a_n| \le \frac{1}{[n]_q}\left[n+\frac{n(n-1)(q+1)}{4}(A-B)\right].$$

Proof. The proof of Theorem 3 follows immediately by using (19).

In its special case, when $A=1$ and $B=-1$, Theorem 3 reduces to the following known results.

Corollary 1 [15]**.** If $f \in \mathrm{K}_q$, then

$$|a_n| \le \frac{1}{[n]_q}\left[n+\frac{n(n-1)(q+1)}{2}\right].$$

If, in Theorem 3, we set $A=1$ and $B=-1$ and let $q \to 1$, we are led to the following known result.

Corollary 2 [18]**.** Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be close-to-convex for $|z|<1$. Then the coefficients satisfy the following inequality:

$$|a_n| \le n|a_1|.$$

In view of Frideman's result [20], there are only nine functions in the class S whose coefficients are rational integers. They are

$$z, \quad \frac{z}{1\pm z}, \quad \frac{z}{1\pm z^2}, \quad \frac{z}{(1\pm z)^2}, \quad \frac{z}{1\pm z+z^2}. \tag{21}$$

We can easily see that the functions in (21) map the unit disk U onto starlike domains. Using the idea of Sahoo et al. [15], we now study special cases of Theorem 3 with respect to the nine functions having integer coefficients. However, in this case it is sufficient to consider the identity function and four other functions which contain the factor $1-z$ instead of $1\pm z$ in the denominator. Especially, Theorem 3 reduces to the following consequences (Theorems 4-8). We provide proofs of the last two consequences (Theorems 7 and 8) as they involve variations in the exponents, though the initial three consequences (Theorems 4-6) follow directly after making exact substitutions for the starlike functions $g(z)$.

Theorem 4. Let $f \in \mathrm{K}_q[A,B]$ be of the form (1) with the Koebe function $g(z)=\frac{z}{(1-z)^2}$. Then for all $n \ge 2$,

$$|a_n| \le \frac{1}{[n]_q}\left[n+\frac{n(n-1)(q+1)}{4}(A-B)\right].$$

Remark 2. If $f \in \mathrm{K}$, with $g(z)=z$, then for all $n \ge 2$, it is well known that

$$|a_n| \le 2.$$

Remark 3. If $f \in \mathrm{K}_q$, with $g(z)=z$, then for all $n \ge 2$, it is well known [21] that

$$|a_n| \le \frac{1-q^2}{1-q^n}.$$

As a generalisation, we have the following result.

Theorem 5. Let $f \in \mathrm{K}_q[A,B]$ be of the form (1) with $g(z)=z$. Then for all $n \ge 2$, we have

$$|a_n| \le \left(\frac{1-q^2}{1-q^n}\right)\cdot\frac{(A-B)}{2}.$$

Remark 4. If $f \in \mathrm{K}$, with $g(z)=\frac{z}{1-z}$, then for all $n \ge 2$, it can be seen that

$$|a_n| \le \frac{(2n-1)}{n}.$$

Remark 5. If $f \in \mathrm{K}_q$, with $g(z)=\frac{z}{1-z}$, then for all $n \ge 2$, it is well known $[21]$ that

$$|a_n| \le \frac{1-q}{1-q^n}[n+q(n-1)].$$

We now state the following analogous result.

Theorem 6. Let $f \in \mathrm{K}_q[A,B]$ be of the form (1) with $g(z)=\frac{z}{1-z}$. Then for all $n \ge 2$,

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$$|a_n| \le \frac{1-q}{1-q^n}\left[n + \frac{q(n-1)}{2}(A-B)\right].$$

Remark 6. If $f \in \mathrm{K}$ with $g(z) = \dfrac{z}{1-z^2}$, then for all $n \ge 2,$ it is known that

$$|a_n| \le \begin{cases} 1 & \text{if} \quad n = 2m-1 \\ 1 & \text{if} \quad n = 2m \end{cases}.$$

As a generalisation, we now state the following theorem along with an outline of its proof.

Theorem 7. Let $f \in \mathrm{K}_q[A,B]$ be the form (1) with $g(z) = \dfrac{z}{1-z^2}$. Then for all $n \ge 2,$ we have

$$|a_n| \le \begin{cases} \dfrac{1}{[n]_q}\left(1 + \dfrac{(q+1)(n-1)}{4}(A-B)\right) & \text{if} \quad n = 2m-1 \\ \\ \dfrac{1}{[n]_q}\left(\dfrac{(q+1)n}{4}(A-B)\right) & \text{if} \quad n = 2m. \end{cases}$$

Proof. Since

$$g(z) = \frac{z}{1-z^2} = \sum_{n=1}^{\infty} z^{2n-1},$$

by $(18),$ we get

$$z + \sum_{n=2}^{\infty}[n]_q\, a_n z^n = \left(\sum_{n=1}^{\infty} z^{2n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right). \tag{22}$$

In order to prove the required optimal bound for $|a_n|$ in this situation, it is appropriate to compare the coefficients of z^{2n-1} and z^{2n} separately. In (22) we first compare the coefficients of z^{2n-1}; for $n \ge 2,$ we have

$$[2n-1]_q\, a_{2n-1} = 1 + \sum_{j=1}^{n-1} p_{2j}.$$

This implies that

$$|a_{2n-1}| \le \frac{1}{[2n-1]_q}\left[1 + \sum_{j=1}^{n-1}|p_{2j}|\right].$$

Using (17), we have

$$|a_{2n-1}| \le \frac{1}{[2n-1]_q}\left[1 + \frac{(1+q)|A-B|}{2}\sum_{j=1}^{n-1} 1\right].$$

Secondly, by comparing the coefficients of z^{2n} , for $n \ge 2,$ we have

$$[2n]_q\, a_{2n} = \sum_{j=1}^{n-1} p_{2j-1},$$

and similarly we have the bound given by

$$|a_{2n}| \le \frac{1}{[2n]_q}\left[\frac{(1+q)(A-B)}{2}\sum_{j=1}^{n-1} 1\right]. \tag{23}$$

We have thus proved the optimal bound for $|a_n|$. In its special case if we let $A=1$ and $B=-1$, we obtain the following known result.

Corollary 3 [15]**.** If $f\in \mathrm{K}_q$ with $g(z)=\dfrac{z}{1-z^2}$, then for all $m\geq 1$,

$$|a_n|\leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{n}{2}(1+q)+\dfrac{1}{2}(1-q)\right), & \text{if} \quad n=2m-1 \\ \left(\dfrac{1-q^2}{1-q^n}\right)\dfrac{n}{2}, & \text{if} \quad n=2m. \end{cases}$$

Remark 7. If $f\in \mathrm{K}$, with $g(z)=\dfrac{z}{1-z+z^2}$, then for all $n\geq 2$, it is known that

$$|a_n|\leq \begin{cases} \dfrac{4n+1}{3n}, & \text{if} \quad n=2m-1 \\ \dfrac{4}{3}, & \text{if} \quad n=2m \\ \dfrac{4n-1}{3n}, & \text{if} \quad n=2m+1. \end{cases}$$

As a generalisation, we have following result.

Theorem 8. Let $f\in \mathrm{K}_q[A,B]$ be of the form (1) with $g(z)=\dfrac{z}{1-z+z^2}$. Then for all $n\geq 2$,

$$|a_n|\leq \begin{cases} \dfrac{1}{3[n]_q}\{(A-B)(1+q)(n+1)-3q\}, & \text{if} \quad n=2m-1 \\ \dfrac{1}{3[n]_q}\{(A-B)(1+q)n\}, & \text{if} \quad n=2m \\ \dfrac{1}{3[n]_q}\{(A-B)(1+q)(n-1)+3\}, & \text{if} \quad n=2m+1. \end{cases}$$

Proof. Since

$$g(z)=\frac{z}{1-z+z^2}=\frac{z(1+z)}{1+z^3}=\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2}+\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1},$$

by (18), we get

$$z+\sum_{n=2}^{\infty}[n]_q a_n z^n=\left(\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2}+\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1}\right)\cdot\left(1+\sum_{n=1}^{\infty}p_n z^n\right). \tag{24}$$

In order to prove the required bonds for $|a_n|$, by first comparing the coefficients of z^{2n-1}, we get

$$[3n-1]_q a_{3n-1} = -(-1)^{n-j} + \sum_{j=1}^{n}(-1)^{n-j} p_{3j-2} + \sum_{j=1}^{n}(-1)^{n-j} p_{3j}. \tag{25}$$

Taking the moduli of both sides in (25) and using (17), for 0<q<1 and $j \geq 1$, we have

$$|a_{3n-1}| \leq \frac{1}{[3n-1]_q}\left((A-B)(1+q)n-q\right).$$

Next, for all $n \geq 1$, if we compare the coefficients of z^{3n} and z^{3n+1} in (24), we obtain the coefficient bounds given, respectively, by

$$|a_{3n}| \leq \frac{1}{[3n]_q}\left((A-B)(1+q)n\right)$$

and

$$|a_{3n+1}| \leq \frac{1}{[3n+1]_q}\left((A-B)(1+q)n+1\right).$$

Hence the required result asserted by Theorem 8.

In its special case when we let $A=1$ and $B=-1$, we have the following known result.

Corollary 4 [15]**.** If $f \in \mathrm{K}_q$ with $g(z) = \dfrac{z}{1-z+z^2}$ then for all $m \geq 1$,

$$|a_n| \leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{1}{3}(2-q)+\dfrac{2n}{3}(1+q)\right), & \text{if} \quad n=2m-1 \\[2ex] \left(\dfrac{1-q^2}{1-q^n}\right)\dfrac{2n}{3}, & \text{if} \quad n=2m \\[2ex] \dfrac{1-q}{1-q^n}\left(\dfrac{2n}{3}(1+q)+\dfrac{1}{3}(1-2q)\right), & \text{if} \quad n=2m+1. \end{cases}$$

PROPERTIES INVOLVING $f(z) = z + \sum_{n=2}^{\infty} X_n z^n$ TO BE IN CLASS $\mathrm{K}_q[A,B]$

In this section we study a number of sufficient conditions for the representation $f(z) = z + \sum_{n=2}^{\infty} X_n z^n$ to be in $\mathrm{K}_q[A,B]$. Rewriting this representation, we get

$$f(z) = z + \sum_{n=2}^{\infty} X_n z^n \qquad (X_0 = 1,\ X_1 = 1). \tag{26}$$

If $f(z)$ is of the form (26), then a simple computation yields

$$(D_q f)z = 1 + \sum_{n=2}^{\infty}[n]_q X_n z^{n-1} \qquad (\forall\, z \in \mathrm{U}).$$

Definition 8. For $t > 0$, let q-gamma function be defined as

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \ \ and \ \ \Gamma_q(1) = 1,$$

where $[t]_q$ is defined in Definition 3.

Considering X_n to be a special function, we consider, in particular, X_n as given by

$$X_n = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)}, \quad \alpha > 0,\ \beta > 0,\ n \in N. \tag{27}$$

Then the function given by (26) reduces to the following form:

$$f(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} z^{n+1},\ \alpha > 0,\ \beta > 0,\ n \in N, \tag{28}$$

which is a normalised q-Mittag-Leffler function. These functions have a wide history and many applications in the field of Geometric Functions Theory, for example in geometric properties including starlikeness, convexity and close-to-convexity for the q-Mittag-Leffler function $f(z)$, which were investigated by Bansal and Prajapat [22] and recently by Srivastava and Bansal [14] and Raza and Din [23]. Differential subordination results associated with the generalised Mittag-Leffler function were also obtained [24]. The q-Mittag-Leffler functions were defined and normalised by Sharma et al. [21]. With this development in view, we now collect a number of sufficient conditions for the functions to be in $K_q[A,B]$.

Theorem 9. Let $f(z)$ be of the form (26) and suppose that

$$\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq \frac{|B-A|}{(B+3)}, \tag{29}$$

where

$$B_n = [n]_q X_n$$

Then $f(z) \in K_q[A,B]$ with $g(z) = \frac{z}{(1-z)}$.

Proof. The proof of Theorem 9 follows easily when we apply (13) in conjunction with

$$g(z) = \frac{z}{(1-z)} \ \ and\ \ (1-z)D_q f(z) = 1 + \sum_{n=1}^{\infty} (B_{n+1} - B_n) z^n. \tag{30}$$

In particular, for the choice of X_n, we have the following result.

Corollary 5. Let $f(z)$ be of the form (28) and suppose that

$$\sum_{n=1}^{\infty} |B_{n+1} - B_n| \leq \frac{|B-A|}{(B+3)},$$

where

$$B_n = [n]_q X_n.$$

Also, let X_n be given by (27). Then $f(z) \in K_q[A,B]$ with $g(z) = \frac{z}{(1-z)}$.

Theorem 10. Let $f(z)$ be of the form (26) and suppose that

$$\sum_{n=1}^{\infty} |B_n - B_{n+1}| \leq \frac{|B-A|}{(B+3)},$$

where

$$B_n = [n+1]_q X_{n+1} - [n]_q X_n.$$

Then $f(z)\in K_q[A,B]$ with $g(z)=\frac{z}{(1-z)^2}$.

Proof. It can be easily seen that

$$g(z)=\frac{z}{(1-z)^2} \quad and \quad (1-z)^2 D_q f(z)=1+(B_1-1)z+\sum_{n=3}^{\infty}(B_{n+1}-B_{n-2})z^{n-1}. \qquad (31)$$

Now by using (13) along with (31), we complete the proof of Theorem 10.

By specialising

$$X_n=\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)}, \quad \alpha>0,\ \beta>0,\ n\in N,$$

we get the following corollary.

Corollary 6. Let $f(z)$ be of the form (28) and suppose that

$$\sum_{n=1}^{\infty}|B_n-B_{n+1}|\leq\frac{|B-A|}{(B+3)},$$

where

$$B_n=[n+1]_q X_{n+1}-[n]_q X_n,$$

and X_n is given by (27). Then $f(z)\in K_q[A,B]$ with $g(z)=\frac{z}{(1-z)^2}$.

Theorem 11. Let $f(z)=z+\sum_{n=2}^{\infty}X_{2n-1}z^{2n-1}$ and assume that

$$\sum_{n=1}^{\infty}|B_{2n-1}-B_{2n+1}|\leq\frac{|B-A|}{(B+3)},$$

where

$$B_n=[n]_q X_n.$$

Then $f(z)\in K_q[A,B]$ with $g(z)=\frac{z}{1-z^2}$.

Proof. The proof of Theorem 11 follows immediately by using (13) and

$$g(z)=\frac{z}{1-z^2} \quad and \quad (1-z^2)D_q f(z)=1-\sum_{n=3}^{\infty}(B_{2n-1}-B_{2n-2})z^{2n}.$$

CONCLUSIONS

We have combined the concept of the familiar Janowski functions and the q-derivative operator and defined a new subclass of q-close-to-convex functions associated with the Janowski functions. Sufficient conditions, de Branges theorem, coefficient inequalities and sufficient conditions for Mittag-Leffler functions to be in the class of Janowski q-close-to-convex functions have been discussed. Relevent connections of our results with those that are already present in the litterature have been pointed out.

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