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Generalisation of certain subclasses of analytic and bi-univalent functions

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Abstract: In this paper we introduce and investigate two new subclasses $\mathbf{H}_{\Sigma}^{q}(\alpha,\lambda)$ and $\mathbf{H}_{\Sigma}^{q}(\beta,\lambda)$ of analytic and bi-univalent functions in the open unit disk U. Furthermore, we find non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

Keywords: analytic functions, univalent functions, bi-univalent functions, coefficient bounds, coefficient estimates, q-derivative operator

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INTRODUCTION

Let $\mathbf{A}$ be the class of all functions f that are analytic in the open unit disk:

$$\mathsf{U} := \{z : z \in \mathrm{C} \quad \text{and} \quad |z| < 1\}$$

and normalised by

$$f(0) = 0 = f'(0) - 1.$$

In other words, the functions f in $\mathbf{A}$ have the Taylor-Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathsf{U}). \tag{1}$$

Furthermore, by $\mathbf{S} \subset \mathbf{A}$ we shall denote the class of all functions which are univalent in U. For two functions f and g which are analytic in U, we say that the function f is subordinate to the function g and write

$$f(z) \prec g(z) \quad (z \in \mathsf{U})$$

if there exists a function

$$w \in \mathbf{B}_0,$$

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where

$$\mathbf{B}_0 = \{w \in \mathbf{A} \ : \ w(0) = 0, |w(z)| < 1 \quad (z \in \mathrm{U})\},$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathrm{U}).$$

If g is univalent in U, then it follows that

$$f(z) \prec g(z) \quad (z \in \mathrm{U}), \Rightarrow f(0) = 0 \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

Moreover, for the functions $f \in \mathbf{A}$ given by (1) and $g \in \mathbf{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathrm{U}),$$

the convolution (or Hadamard product) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{2}$$

We next denote by $\mathbf{P}$ the class of analytic functions p which are normalised by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3}$$

such that

$$\mathrm{Re}\, p(z) > 0.$$

Furthermore, it is well known that every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathrm{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots. \tag{4}$$

A function $f \in \mathbf{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote the class of all such functions by Σ. The pioneering work of Srivastava et al. [1] actually revived the study of bi-univalent functions in recent years. In a substantially large number of work subsequent to the work of Srivastava et al. [1], several distinct subclasses of the bi-univalent function class were presented and examined similarly by many authors. For example, the function classes $H_{\Sigma}(\tau, \mu, \lambda, \delta; \alpha)$ and $H_{\Sigma}(\tau, \mu, \lambda, \gamma; \beta)$ were defined and the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were obtained by Srivastava et al [2]. The upper bounds for the second Hankel determinant for certain subclasses of analytic and bi-univalent functions were obtained by Caglar et al [3]. Several new subclasses of the class of m-fold symmetric bi-univalent functions were introduced and the initial estimates of the Taylor-Maclaurin series as well as some Fekete-Szegö functional problems for each of their defined function classes were obtained by Tang et al. [4] and Srivastava et al [5]. Several other well-known mathematicians gave their findings on this subject [e.g. 6-16].

We now recall some basic definitions and concept details of the q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$ and

$$\mathbb{N} = \{1, 2, 3 \ldots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, 3 \ldots\}).$$

Definition 1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 2. Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

Definition 3 [17,18]**.** The q-derivative (or q-difference) D_q of a function f is defined in a given subset of $\mathbb{C}$ by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ f'(0) & (z = 0). \end{cases} \tag{5}$$

We note from Definition 3 that the difference operator $D_q f$ converges to the ordinary differential operator:

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(qz) - f(z)}{(1-q)z} = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It is readily deduced from (1) and (5) that

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} a_n [n]_q z^{n-1}.$$

In Geometric Function Theory several subclasses belonging to the class of normalised analytic functions $\mathbf{A}$ have been examined already. The q-calculus defined above provides significant tools that have been widely used for investigating several subclasses of $\mathbf{A}$. Ismail et al. [19] were the first to use the q-derivative operator D_q in order to study a certain q-analogue of the class S^* of starlike functions in $\mathbb{U}$. In fact, historically speaking, a remarkably significant usage of the q-calculus in the context of Geometric Function Theory of Complex Analysis was basically furnished and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava [20]. See also Srivastava and Bansal [21].

Motivated by the work of Frasin [12], Bulut [22] and the above-mentioned work, we here introduce two new subclasses of the function class Σ and find non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

In order to derive our main results, the following Lemma will be required.

Lemma 1 [23]. If $p \in \mathbf{P}$, then $|p_k| \le 2$ for each k, where $\mathbf{P}$ is the family of all functions p analytic in $\mathbb{U}$ for which

$$\mathrm{Re}(p(z)) > 0 \quad \text{and} \quad p(z) = 1 + p_1 z + p_2 z^2 + \ldots$$

for $z \in \mathbb{U}$.

Throughout in this paper, we assume that

$$0 < \beta \le 1 \quad 0 < \alpha \le 1 \quad \text{and} \quad \lambda \ge 0.$$

COEFFICIENT BOUNDS FOR FUNCTION CLASS $\mathbf{H}_{\Sigma}^{q}(\alpha,\lambda)$

Definition 4. A function $f \in \mathbf{A}$ of the form given by (1) is in the function class $\mathbf{H}_{\Sigma}^{q}(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(D_q f(z) + z\lambda D_q\left(D_q f(z)\right)\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathbb{U}) \tag{6}$$

and

$$\left|\arg\left(D_q g(w) + w\lambda D_q\left(D_q g(w)\right)\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathbb{U}), \tag{7}$$

where function g is given by (4).

Remark 1. Firstly, it is easily seen that

$$\lim_{q \to 1^-} \mathbf{H}_{\Sigma}^{q}(\alpha,\lambda) = \mathbf{H}_{\Sigma}(\alpha,\lambda),$$

where $\mathbf{H}_{\Sigma}(\alpha,\lambda)$ is the function class introduced and studied by Frasin [12]. Secondly, we have

$$\lim_{q \to 1^-} \mathbf{H}_{\Sigma}^{q}(\alpha,0) = \mathbf{H}_{\Sigma}(\alpha) = \mathbf{H}_{\Sigma}^{\alpha},$$

where $\mathbf{H}_{\Sigma}^{\alpha}$ is the function class introduced and studied by Srivastava et al. [13]. Thirdly,

$$\mathbf{H}_{\Sigma}^{q}(\alpha,0) = \mathbf{H}_{\Sigma}^{q}(\alpha) = \mathbf{H}_{\Sigma}^{q,\alpha},$$

where $\mathbf{H}_{\Sigma}^{q,\alpha}$ is the function class introduced and studied by Bulut [22].

Theorem 1. Let the function $f \in \mathbf{A}$ of the form given by (1) be in the function class $\mathbf{H}_{\Sigma}^{q}(\alpha,\lambda)$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{2[3]_q\left(1+\lambda[2]_q\right)\alpha - [2]_q^2(1+\lambda)^2(\alpha-1)}} \tag{8}$$

and

$$|a_3| \le \frac{4\alpha^2}{[2]_q^2(1+\lambda)^2} + \frac{2\alpha}{[3]_q\left(1+[2]_q\lambda\right)}. \tag{9}$$

Proof. It can be seen from conditions (6) and (7) that

$$D_q f(z) + z\lambda D_q\left(D_q f(z)\right) = [P(z)]^{\alpha} \tag{10}$$

and

$$D_q g(w) + w\lambda D_q\left(D_q g(w)\right) = [Q(w)]^{\alpha}, \tag{11}$$

where

$$P(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad \text{and} \quad Q(w) = 1 + q_1 w + q_2 w^2 + \ldots$$

in $\mathbf{P}$. Now equating the coefficients in (10) and (11), we have

$$[2]_q(1+\lambda)a_2 = \alpha p_1, \tag{12}$$

$$[3]_q\left(1+\lambda[2]_q\right)a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{13}$$

$$-[2]_q(1+\lambda)a_2=\alpha q_1 \tag{14}$$

and

$$[3]_q\left(1+\lambda[2]_q\right)\left(2a_2^2-a_3\right)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{15}$$

From (12) and (14), we have

$$p_1=-q_1 \tag{16}$$

and

$$2[2]_q^2(1+\lambda)^2a_2^2=\alpha^2\left(p_1^2+q_1^2\right). \tag{17}$$

Also, from (13), (15) and (17), we find, after some simplification, that

$$a_2^2=\frac{\alpha^2}{\left(2[3]_q\left(1+\lambda[2]_q\right)\right)\alpha-[2]_q^2(1+\lambda)^2(1-\alpha)}(p_2+q_2). \tag{18}$$

Finally, by applying Lemma 1 in conjunction with (18), we obtain the desired estimate on the coefficient $|a_2|$ as stated in (8). Next, in order to prove (9), we subtract (15) from (13). Indeed, we find that

$$2[3]_q\left(1+\lambda[2]_q\right)a_3-2[3]_q\left(1+\lambda[2]_q\right)a_2^2=\alpha\left(p_2-q_2\right)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2-q_1^2\right). \tag{19}$$

It follows from (16), (17) and (19) that

$$a_3=\frac{\alpha^2\left(p_1^2+q_1^2\right)}{2[2]_q^2(1+\lambda)^2}+\frac{\alpha\left(p_2-q_2\right)}{2[3]_q\left(1+\lambda[2]_q\right)}. \tag{20}$$

Finally, by using Lemma 1 and (20), we find the desired estimate on the coefficient $|a_3|$ as stated in (9).

Remark 2. By substituting $\lambda=0$ in Theorem 1, we obtain the coefficient bounds for $|a_2|$ and $|a_3|$ given by Bulut [22]. Then by putting $\lambda=0$ and letting $q\to 1^-$, we have the following known result.

Corollary 1 [13]**.** Let function $f(z)$ given by the Taylor-Maclaurin series expansion (1) be in the class $\mathbf{H}_\Sigma^\alpha$ $(0\le\alpha\le 1)$. Then

$$|a_2|\le\alpha\sqrt{\frac{2}{\alpha+2}}\quad\text{and}\quad |a_3|\le\frac{\alpha(3\alpha+2)}{3}.$$

Theorem 2. Let function $f\in\mathbf{H}_\Sigma^q(\alpha,\lambda)$ and be of the form given by (1). Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}4|h(\mu)| & |\mu-1|\ge\left|1-\frac{[2]_q^2(1+\lambda)^2(1-\alpha)}{2[3]_q\left(1+\lambda[2]_q\right)\alpha}\right|,\\[2ex] \frac{2\alpha}{[3]_q\left(1+\lambda[2]_q\right)} & |\mu-1|\le\left|1-\frac{[2]_q^2(1+\lambda)^2(1-\alpha)}{2[3]_q\left(1+\lambda[2]_q\right)\alpha}\right|\end{cases} \tag{21}$$

where

$$h(\mu)=\frac{(\mu-1)\alpha^2}{2[3]_q\left(1+\lambda[2]_q\right)\alpha-[2]_q^2(1+\lambda)^2(1-\alpha)}. \tag{22}$$

Proof. We can show that the inequalities in (21) hold true for $f \in \mathbf{H}_{\Sigma}^{q}(\alpha,\lambda)$. After some straightforward simplification of (17), (18) and (19), the following is obtained :

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{\alpha}{2[3]_q\left(1+\lambda[2]_q\right)} \right) p_2 + \left(h(\mu) - \frac{\alpha}{2[3]_q\left(1+\lambda[2]_q\right)} \right) q_2, \tag{23}$$

where $h(\mu)$ is given by (22). From (23), we now conclude the assertion of our Theorem.

COEFFICIENT BOUNDS FOR FUNCTION CLASS $\mathbf{H}_{\Sigma}^{q}(\beta,\lambda)$

Definition 5. A function $f \in \mathbf{A}$ of the form given by (1) is in the function class $\mathbf{H}_{\Sigma}^{q}(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \mathrm{Re}\left(D_q f(z) + z\lambda D_q\left(D_q f(z)\right)\right) > \beta \quad (z \in \mathrm{U}) \tag{24}$$

and

$$\mathrm{Re}\left(D_q g(w) + w\lambda D_q\left(D_q g(w)\right)\right) > \beta \quad (w \in \mathrm{U}), \tag{25}$$

where function g is define by (4).

Remark 3. Firstly, it is readily observed that

$$\lim_{q \to 1^-} \mathbf{H}_{\Sigma}^{q}(\beta,\lambda) = \mathbf{H}_{\Sigma}(\beta,\lambda),$$

where $\mathbf{H}_{\Sigma}(\beta,\lambda)$ is the function class introduced and studied by Frasin [12]. Secondly, we have

$$\lim_{q \to 1^-} \mathbf{H}_{\Sigma}^{q}(\beta,0) = \mathbf{H}_{\Sigma}(\beta),$$

where $\mathbf{H}_{\Sigma}(\beta)$ is the function class introduced and studied by Srivastava et al. [13]. Thirdly,

$$\mathbf{H}_{\Sigma}^{q}(\beta,0) = \mathbf{H}_{\Sigma}^{q}(\beta),$$

where $\mathbf{H}_{\Sigma}^{q}(\beta)$ is the function class introduced and studied by Bulut [22].

Theorem 3. Let the function $f \in \mathbf{A}$ of the form given by (1) be in the function class $\mathbf{H}_{\Sigma}^{q}(\beta,\lambda)$. Then

$$|a_2| \le \min\left(\frac{2(1-\beta)}{[2]_q(1+\lambda)}, \sqrt{\frac{2(1-\beta)}{[3]_q\left(1+\lambda[2]_q\right)}} \right) \tag{26}$$

and

$$|a_3| \le \frac{2(1-\beta)}{[3]_q\left(1+\lambda[2]_q\right)}. \tag{27}$$

Proof. Firstly, it follows from conditions (24) and (25) that

$$D_q f(z) + z\lambda D_q\left(D_q f(z)\right) = \beta + (1-\beta)P(z) \quad (z \in \mathrm{U}) \tag{28}$$

and

$$D_q g(w) + w\lambda D_q\left(D_q g(w)\right) = \beta + (1-\beta)Q(w) \quad (w \in \mathrm{U}), \tag{29}$$

where

$$P(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad \text{and} \quad Q(w) = 1 + q_1 w + q_2 w^2 + \ldots$$

in $\mathbf{P}$. Now equating the coefficients in (28) and (29), we have

$$[2]_q(1+\lambda)a_2 = (1-\beta)p_1, \tag{30}$$

$$[3]_q\left(1+\lambda[2]_q\right)a_3 = (1-\beta)p_2, \tag{31}$$

$$-[2]_q(1+\lambda)a_2 = (1-\beta)q_1 \tag{32}$$

and

$$[3]_q(1+\lambda[2]_q)(2a_2^2-a_3)=(1-\beta)q_2. \tag{33}$$

From (30) and (32) we have

$$p_1=-q_1 \tag{34}$$

and

$$2[2]_q^2(1+\lambda)^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \tag{35}$$

Also, from (31) and (33) we have

$$2[3]_q(1+\lambda[2]_q)a_2^2=(1-\beta)(p_2+q_2). \tag{36}$$

Finally, by applying Lemma 1 in conjunction with (35) and (36), we obtain the desired estimate on the coefficient $|a_2|$ as stated in (26).

Next, in order to prove (27) we subtract (33) from (31). We have

$$2[3]_q(1+\lambda[2]_q)a_3-2[3]_q(1+\lambda[2]_q)a_2^2=(1-\beta)(p_2+q_2), \tag{37}$$

which, upon substitution of the value of a_2^2 from (35), yields

$$a_3=\frac{(1-\beta)(p_1^2+q_1^2)}{2[2]_q^2(1+\lambda)^2}+\frac{(1-\beta)(p_2-q_2)}{2[3]_q(1+\lambda[2]_q)}. \tag{38}$$

On the other hand, by using the equation (36) on (37), we have

$$a_3=\frac{(1-\beta)(p_2)}{2[3]_q(1+\lambda[2]_q)}. \tag{39}$$

Finally, by applying Lemma 1 to (38) and (39), we obtain the desired estimate on the coefficient $|a_3|$ as stated in (27).

Taking $\lambda=0,$ we obtain the following known result.

Corollary 2 [22]**.** Let the function $f\in\mathbf{A}$ of the form given by (1) be in the function class $\mathbf{H}_\Sigma^q(\beta)$. Then

$$|a_2|\le\min\left(\frac{2(1-\beta)}{[2]_q},\sqrt{\frac{2(1-\beta)}{[3]_q}}\right)$$

and

$$|a_3|\le\frac{2(1-\beta)}{[3]_q}.$$

CONCLUSIONS

This research work presents some properties of certain new subclasses of analytic and bi-univalent functions in the open unit disk U. Coefficient estimates for these newly function classes have been discussed. Also, we have pointed out some known results deduced from our main results.

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