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Full Paper

A new study on generalised absolute matrix summability methods

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Abstract: A theorem on $\varphi-|A;\delta|_k$ summability of infinite series, which generalises the result dealing with $|\bar{N},p_n|_k$ summability of infinite series, has been proved. This theorem also contains some new results related to the $|A,p_n|_k$ and $|C,1|_k$ summability methods for the special cases of δ, (p_n), (φ_n), and (a_{nv}).

Keywords: Riesz mean, summability factor, absolute matrix summability, almost increasing sequences, infinite series, Hölder inequality, Minkowski inequality

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INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$ [1]. Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking an example, viz. $b_n = ne^{(-1)^n}$.

Let $\sum_{v=0}^{\infty} a_v$ be an infinite numerical series with its partial sums $s_n = \sum_{v=0}^{n} a_v$. Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \ (P_{-i} = p_{-i} = 0, \ i \ge 1). \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (σ_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) [2]. The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \ge 1$, if [3]

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |\sigma_n-\sigma_{n-1}|^k<\infty. \tag{3}$$

Let $A=(a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v, \quad n=0,1,... \tag{4}$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi-|A;\delta|_k$, $k\geq 1$ and $\delta\geq 0$, if [4]

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left|\bar{\Delta}A_n(s)\right|^k<\infty, \tag{5}$$

where

$$\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s). \tag{6}$$

If we take $\varphi_n=P_n/p_n$, then $\varphi-|A;\delta|_k$ summability reduces to $|A,p_n;\delta|_k$ summability [5]. For $\delta=0$ and $\varphi_n=P_n/p_n$, $\varphi-|A;\delta|_k$ summability reduces to $|A,p_n|_k$ summability [6]. Also, if we take $\delta=0$ and $\varphi_n=n$ for all n, then $\varphi-|A;\delta|_k$ summability reduces to $|A|_k$ summability [7]. Additionally, when we take $\delta=0$, $\varphi_n=P_n/p_n$ and $a_{nv}=p_v/P_n$, then we get $|\bar{N},p_n|_k$ summability. Furthermore, by taking $\delta=0$, $\varphi_n=n$, $a_{nv}=p_v/P_n$ and $p_n=1$ for all values of n, we get $|C,1|_k$ summability [8].

KNOWN RESULT

Bor [9] has proved the following theorem for $|\bar{N},p_n|_k$ summability factors of infinite series.

Theorem 1. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$|\Delta\lambda_n|\leq\beta_n, \tag{7}$$

$$\beta_n\to 0 \ \ as \ \ n\to\infty, \tag{8}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n<\infty, \tag{9}$$

$$|\lambda_n|X_n=O(1). \tag{10}$$

If

$$\sum_{v=1}^{n}\frac{|s_v|^k}{v}=O(X_n) \ \ as \ \ n\to\infty \tag{11}$$

and (p_n) is a sequence such that

$$P_n=O(np_n), \tag{12}$$

$$P_n\Delta p_n=O(p_np_{n+1}), \tag{13}$$

then the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$ is summable $|\bar{N},p_n|_k$, $k\geq 1$, where $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$.

Remark. It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ [10].

MAIN RESULTS

The aim of this paper is to generalise Theorem 1 for $\varphi-\left|A;\delta\right|_k$ summability method. Before stating the main theorem, we must first introduce some further notations. Given a normal matrix $A=\left(a_{nv}\right)$, we associate two lower semimatrices $\overline{A}=(\overline{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ as follows:

$$\overline{a}_{nv}=\sum_{i=v}^{n} a_{ni}\,,\quad n,v=0,1,... \tag{14}$$

and

$$\hat{a}_{00}=\overline{a}_{00}=a_{00}\,,\quad \hat{a}_{nv}=\overline{a}_{nv}-\overline{a}_{n-1,v}\,,\quad n=1,2,... \tag{15}$$

It may be noted that $\overline{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations respectively. Then we have

$$A_n\left(s\right)=\sum_{v=0}^{n} a_{nv}s_v=\sum_{v=0}^{n}\overline{a}_{nv}a_v \tag{16}$$

and

$$\overline{\Delta}A_n\left(s\right)=\sum_{v=0}^{n}\hat{a}_{nv}a_v\,. \tag{17}$$

Now the following theorem shall be proved.

Theorem 2. Let $A=(a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0}=1\,,\quad n=0,1,..., \tag{18}$$

$$a_{n-1,v}\geq a_{nv}\ \text{ for }\ n\geq v+1, \tag{19}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{20}$$

$$\left|\hat{a}_{n,v+1}\right|=O\left(v\left|\Delta_v(\hat{a}_{nv})\right|\right), \tag{21}$$

$$\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}\left|\Delta_v(\hat{a}_{nv})\right|=O\left(\varphi_v^{\delta k}\frac{p_v}{P_v}\right)\quad as\quad m\to\infty, \tag{22}$$

$$\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}\left|\hat{a}_{n,v+1}\right|=O\left(\varphi_v^{\delta k}\right)\quad as\quad m\to\infty\,. \tag{23}$$

Let (X_n) be an almost increasing sequence, $\varphi_n p_n=O\left(P_n\right)$ and $\left|\lambda_n\right|=O\left(\frac{1}{X_n}\right)=O(1)$. If conditions (7)-(9) and (12)-(13) of Theorem 1 and

$$\sum_{v=1}^{n}\varphi_v^{\delta k}\frac{\left|s_v\right|^k}{v}=O(X_n)\quad as\quad n\to\infty \tag{24}$$

are satisfied, then the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$ is summable $\varphi-\left|A;\delta\right|_k$, $k\geq1$ and $0\leq\delta<1/k$, where $\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}$.

It should be noted that if we take $\delta=0$, $\varphi_n = P_n/p_n$ and $a_{nv} = p_v/P_n$, then we get Theorem 1. In this case condition (24) reduces to condition (11). Also, conditions (18)-(23) are automatically satisfied.

We need the following lemmas for proof of Theorem 2.

Lemma 1 [11]. If (X_n) is an almost increasing sequence, then under conditions (8)-(9), we have

$$nX_n\beta_n = O(1) \ as \ \ n\to\infty, \tag{25}$$

$$\sum_{n=1}^{\infty}\beta_n X_n < \infty . \tag{26}$$

Lemma 2 [12]. If conditions (12) and (13) are satisfied, then we have

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{27}$$

Proof of Theorem 2

Let (I_n) denote the A-transform of the series $\sum \frac{a_n\lambda_n P_n}{np_n}$. Then by (16) and (17), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv}\frac{a_v\lambda_v P_v}{vp_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}\lambda_v P_v}{vp_v}\right)\sum_{r=1}^{v}a_r + \frac{\hat{a}_{nn}P_n\lambda_n}{np_n}\sum_{v=1}^{n}a_v \\
&= \sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}\lambda_v P_v}{vp_v}\right)s_v + \frac{\hat{a}_{nn}P_n\lambda_n}{np_n}s_n \\
&= \frac{a_{nn}P_n\lambda_n}{np_n}s_n + \sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v + \sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_{v+1}}{(v+1)p_{v+1}}s_v + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_v\Delta\left(\frac{P_v}{vp_v}\right)s_v \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 2, it is sufficient to show that

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left|I_{n,r}\right|^k < \infty, \text{ for } r = 1,2,3,4.$$

First, by using Abel's transformation, we have

$$\begin{aligned}
\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|I_{n,1}\right|^k &= \sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|\frac{a_{nn}P_n\lambda_n}{np_n}s_n\right|^k = O(1)\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k\frac{1}{n^k}\left(\frac{P_n}{p_n}\right)^k |\lambda_n|^{k-1}|\lambda_n||s_n|^k \\
&= O(1)\sum_{n=1}^{m}\varphi_n^{\delta k}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{1}{n^k}|\lambda_n||s_n|^k = O(1)\sum_{n=1}^{m}\varphi_n^{\delta k}|\lambda_n|\frac{|s_n|^k}{n} \\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\varphi_r^{\delta k}\frac{|s_r|^k}{r} + O(1)|\lambda_m|\sum_{n=1}^{m}\varphi_n^{\delta k}\frac{|s_n|^k}{n}
\end{aligned}$$

Maejo Int. J. Sci. Technol. **2018**, *12*(03), 199-205

$$= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n + O(1)|\lambda_m|X_m$$

$$= O(1)\sum_{n=1}^{m-1}\beta_n X_n + O(1)|\lambda_m|X_m = O(1) \quad as \quad m\to\infty,$$

in view of (7), (10), (12), (20), (24) and (26).

Applying Hölder's inequality with indices k and k', where $k>1$ and $1/k+1/k'=1,$ as in $I_{n,1}$, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v\right|^k \le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{P_v}{vp_v}|\Delta_v(\hat{a}_{nv})||\lambda_v||s_v|\right\}^k$$

$$\le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k \times\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.$$

Now using (14), (15) and (19), we get

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1}|a_{nv}-a_{n-1,v}| = \sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv}) = \overline{a}_{n-1,0}-a_{n-1,0}-\overline{a}_{n0}+a_{n0}+a_{nn}$$

$$= 1-a_{n-1,0}-1+a_{n0}+a_{nn}$$

$$= a_{n0}-a_{n-1,0}+a_{nn}$$

$$\le a_{nn}. \tag{28}$$

Hence

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,2}|^k \le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|s_v|^k\right\}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\Delta_v(\hat{a}_{nv})|$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{p_v}{P_v}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\lambda_v|^{k-1}|\lambda_v||s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}|\lambda_v|\frac{|s_v|^k}{v} = O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Now using the fact that $P_{v+1}=O((v+1)p_{v+1})$ by (12), and Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}|I_{n,3}|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_{v+1}}{(v+1)p_{v+1}}s_v\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|^k\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right\}^{k-1}.$$

Using (14), (15) and (19), we have

$$\hat{a}_{n,v+1} = \overline{a}_{n,v+1}-\overline{a}_{n-1,v+1} = \sum_{i=v+1}^{n}a_{ni}-\sum_{i=v+1}^{n-1}a_{n-1,i} = a_{nn}+\sum_{i=v+1}^{n-1}(a_{ni}-a_{n-1,i}) \le a_{nn}.$$

Then

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|I_{n,3}\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v\left|s_v\right|^k$$

$$= O(1)\sum_{v=1}^{m}\beta_v|s_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\hat{a}_{n,v+1}| = O(1)\sum_{v=1}^{m}\varphi_v^{\delta k} v\beta_v\frac{\left|s_v\right|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{\left|s_r\right|^k}{r} + O(1)m\beta_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{\left|s_v\right|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\left|\Delta(v\beta_v)\right|X_v + O(1)m\beta_m X_m$$

$$= O(1)\sum_{v=1}^{m-1}v\left|\Delta\beta_v\right|X_v + O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)m\beta_m X_m = O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 2 and Lemma 1.

Now since $\Delta\left(\frac{P_v}{vp_v}\right) = O\left(\frac{1}{v}\right)$ by Lemma 2, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|I_{n,4}\right|^k = \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_v\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v||s_v|\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k\times\left\{\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\right\}^{k-1}.$$

By using (28), we get

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|I_{n,4}\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k;$$

then by (20), we obtain

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|I_{n,4}\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_v|^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_v|^{k-1}|\lambda_v|\left|s_v\right|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}|\hat{a}_{n,v+1}| = O(1)\sum_{v=1}^{m}\varphi_v^{\delta k}\left|\lambda_v\right|\frac{\left|s_v\right|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_v\right|\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{\left|s_r\right|^k}{r} + O(1)\left|\lambda_m\right|\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{\left|s_v\right|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\left|\Delta\lambda_v\right|X_v + O(1)\left|\lambda_m\right|X_m$$

$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)\left|\lambda_m\right|X_m = O(1) \quad as \quad m\to\infty,$$

by (7), (10), (23), (24) and (26). This completes the proof of Theorem 2.

CONCLUSIONS

In this paper I have proved a main theorem dealing with a general absolute matrix summability method of factored infinite series. A new result can be obtained for the $\left|A, p_n\right|_k$ summability method by taking $\delta = 0$ and $\varphi_n = P_n/p_n$. Also, if we take $\delta = 0$, $\varphi_n = n$, $a_{nv} = p_v/P_n$

Maejo Int. J. Sci. Technol. **2018**, *12*(03), 199-205

and $p_n = 1$ for all n in Theorem 2, then we get another new result dealing with the $\left|C,1\right|_k$ summability method.

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