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***Full Paper***

# Temporal intuitionistic fuzzy metric spaces

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**Abstract:** In this study temporal intuitionistic fuzzy metric space is defined to obtain a dynamic measure that expresses distances between the spatio-temporal points whose positions change over time and also between the data represented by these points. In order to define this new approach, the concepts of temporal fuzzy t-norm, temporal fuzzy t-conorm and temporal fuzzy negation, which do not exist in the literature, are defined and some basic features of these concepts are examined. The concept of temporal intuitionistic fuzzy metric spaces is defined with a new approach on the basis of the idea that the degrees of nearness and non-nearness change over time. On the other hand, the fundamental topological properties of temporal intuitionistic fuzzy metric space are also examined. We show that the fundamental properties provided by classical and fuzzy metric spaces are also preserved by this new temporal metric space. Thus, a new and more general and more dynamic metric topology is obtained, in which the basic topological properties of fuzzy and intuitionistic fuzzy metric spaces are preserved.

**Keywords:** fuzzy sets, fuzzy metric spaces, intuitionistic fuzzy metric spaces, temporal intuitionistic sets, temporal intuitionistic spaces

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## INTRODUCTION

After Zadeh [1] gave a different direction to mathematics by first defining the concept of fuzzy logic and consequently the fuzzy set in 1965, the need to reconsider all concepts by fuzzy set theory arose. With the new approaches introduced by the fuzzy sets, the concept of distance has been reconsidered with many different approaches. One of the best known of these is fuzzy metric defined by Kramosil and Michalek [2] in 1975. In this metric, the distance between points is expressed with a new concept called the degree of nearness. This fuzzy metric space was modified by George and Veeramani [3] to get Hausdorff topology.

The fuzzy set theory has been extended by many different approaches. One of them is the intuitionistic fuzzy set theory defined by Atanassov [4]. In intuitionistic fuzzy set theory the degree of non-membership and uncertainty are defined besides the degree of membership. The intuitionistic fuzzy set theory has been proven to be a highly effective method in dealing with bipolar situations in many studies in different fields recently. [5-10]. As in the fuzzy set theory, in the intuitionistic fuzzy set theory mathematical concepts are redefined in more effective ways. Especially by using negative information and uncertainty when defining concepts, it has made it possible to obtain wider and realistic mathematical models. The concept of intuitionistic fuzzy metric space, the one of these models, was defined by Park [11] with aspect of the fuzzy metric defined by George and Veermani [3]. In this definition in order to measure the distance between points with intuitionistic fuzzy metric, besides the degree of nearness, the degree of non-nearness and uncertainty are also used. The topological properties of intuitionistic fuzzy metric spaces have been investigated in many studies [7, 20-23, 25].

Another important extension of fuzzy sets is the temporal intuitionistic fuzzy set obtained by adding the time parameter to the intuitionistic fuzzy set [12]. It is obvious that the idea of changing the degree of membership and degree of non-membership over time and location has created a rich field of study in spatio-temporal research fields such as weather, economy, image processing and video processing. On the other hand, effective results have been obtained in many different ways on the mentioned subjects [13-17]. The fact that the concept of the temporal intuitionistic fuzzy metric space to be obtained by taking into account the time parameter is still not defined in the literature constitutes a great deficiency.

The main motivation of this study is the idea of dynamising the distance measures used in dynamic areas by defining them with spatio-temporal dynamics. The positions of the points that we want to measure the distance between them may change over time, and this fact emphasises that the concept of metric we use when measuring the distance must be connected to the concept of time as well as positions of the points. Another important point is the advantages of this mathematical structure, which we use in measuring the distance, in having an adaptive and variable structure. So the concept of temporal intuitionistic fuzzy metric that is defined by adding time parameter will be a more general and adaptive metric. In our study temporal intuitionistic fuzzy metric spaces are defined and their basic topological properties are examined. The main motivation of the study is to introduce a new metric concept that can measure the distance between moving points and sets instantly and also change adaptively over time. In the following sections the basic definitions used in the study are given and the temporal intuitionistic fuzzy metric space is defined and its properties are examined.

## PRELIMINARIES

**Definition 1.** A fuzzy set $A$ on a non-empty set $X$ is given by a set of ordered pairs such that $A=\{(x,\mu_A(x));\, x\in X\}$ where $\mu_A : X \to I=[0,1]$. For $x\in X$, $\mu_A(x)$ represents the degree of membership of $x$ to the fuzzy set $A$ and $\mu_A$ function is also called a membership function [1]. By $FS^X$, we denote to the set of all fuzzy sets.

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Negation, triangular norm (t-norm) and triangular conorm (t-conorm) operators, which generalise the basic operations of complementation, intersection and union in set theory, are also used in the concept of fuzzy metric. Negation, triangular norm (t-norm) and triangular conorm (t-conorm) operators are defined as follows.

**Definition 2.** If $N : I \to I$ function satisfies the following conditions, $N$ is called a negation:

i. $N(0) = 1$ , $N(1) = 0$;

ii. $N$ is a non-increasing function. ( $x \leq y \Rightarrow N(x) \geq N(y)$ ).

If a negation is monotonously decreasing $(x < y \Rightarrow N(x) > N(y))$ and continuous, it is called a strict negation. If a strict negation $N$ is an involution ( $N(N(x)) = x$ for all $x \in X$ ), it is called a strong negation [18].

**Definition 3.** The negation of a fuzzy set $A$ for $A \in FS^X$ is defined by $N(A) = \{(x, \mu_{N(A)}(x)) : x \in X\} = \{(x, N(\mu_A(x))) : x \in X\}$ [18].

**Definition 4.** If the binary operation $T : I \times I \to I$ satisfies the following conditions, $T : I \times I \to I$ is called a $t-$norm:

i. $T(0,0) = 0$, $T(1,a) = a$ (bounded condition or closure property);

ii. $T(a,b) \leq T(c,d)$ if $a \leq c$ and $b \leq d$ (monotonicity);

iii. $T(a,b) = T(b,a)$ (commutativity);

iv. $T(a,T(b,c)) = T(T(a,b),c)$ (associativity) [18].

**Definition 5.** If the binary operation $T : I \times I \to I$ satisfies the following conditions, $T : I \times I \to I$ is called a $t-$conorm or $s$-norm:

i. $S(1,1) = 1$, $S(0,a) = a$ (bounded condition);

ii. $S(a,b) \leq S(c,d)$ if $a \leq c$ and $b \leq d$ (monotonicity);

iii. $S(a,b) = S(b,a)$ (commutativity);

iv. $S(a,S(b,c)) = S(S(a,b),c)$ (associativity) [18].

**Definition 6.** The ordered triplet $(S,T,N)$ where $T$ is a t-norm, $S$ is an s-norm and $N$ is a strong negation is called the De Morgan Triplet if it satisfies the condition $S(x,y) = N(T(N(x),N(y)))$ for $\forall x,y \in [0,1]$ [18].

**Remark 1.** If $(S,T,N)$ is a De Morgan Triplet, then $T(x,y) = N(S(N(x),N(y)))$ [18].

There are many extensions of fuzzy sets in the literature. One of the most used of these is intuitionistic fuzzy set defined by Atanassov. Intuitionistic fuzzy sets, which differ from fuzzy sets on the concepts of uncertainty and non-membership, are used in many different areas.

**Definition 7.** An intuitionistic fuzzy set in a non-empty set $X$ given by a set of ordered triplets

$$A = \{(x, \mu_A(x), \eta_A(x)) : x \in X\},$$

where $\mu_A : X \to I$, $\eta_A : X \to I$ and $I = [0,1]$, is a function such that $0 \leq \mu_A(x) + \eta_A(x) \leq 1$ for all $x \in X$. For $x \in X$, $\mu_A(x)$ and $\eta_A(x)$ represent the degree of membership and degree of non-

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membership of $x$ to $A$ respectively. For each $x \in X$, the intuitionistic fuzzy index of $x$ in $A$ can be defined as $\pi_A(x) = 1 - \mu_A(x) - \eta_A(x)$. $\pi_A(x)$ is the so-called degree of hesitation or indeterminacy [19]. Atanassov [19] also defined the concept of temporal intuitionistic fuzzy sets.

**Definition 8.** Let $X$ be a universe and $T$ be a non-empty time set. We call the elements of $T$ 'time moments'. Based on the definition of intuitionistic fuzzy set, a temporal intuitionistic fuzzy set is defined as the following:

$$A(T) = \{(x, \mu_A(x,t), \eta_A(x,t)) : (x,t) \in X \times T\},$$

where

i. $A \subseteq X$ is fixed;

ii. $\mu_A(x,t) + \eta_A(x,t) \le 1$ $\mu_A(x,t) + \eta_A(x,t) \le 1$ for every $(x,t) \in X \times T$;

iii. $\mu_A(x,t)$ and $\eta_A(x,t)$ are the degrees of membership and non-membership, respectively, of the element at $x \in X$ at the time moment $t \in T$ [19].

Obviously, every intuitionistic fuzzy set can be expressed as temporal intuitionistic fuzzy set via a singular time set. In addition, all operations and operators defined for intuitionistic fuzzy sets can be defined for temporal intuitionistic fuzzy sets.

The concept of distance, one of the most fundamental concepts of mathematics and engineering, allows us to make comparisons among points or situations by measuring the distance between two numbers or states. The fuzzy set theory contributes to defining new methods in dealing with the concept of distance with new approaches which cannot be defined in the classic set.

The concept of distance in fuzzy set theory has been defined in many different ways. One of them is the intuitionistic fuzzy metric space defined by Park [11]:

**Definition 9.** A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary (non-empty) set, * is a continuous $t-$norm, $\Diamond$ a continuous $t-$conorm and $M$ , $N$ are fuzzy sets on $X \times X \times (0,\infty)$, satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$:

**a.** $M(x,y,t) + N(x,y,t) \le 1$;

**b.** $M(x,y,t) > 0$;

**c.** $M(x,y,t) = 1$ if and only if $x = y$;

**d.** $M(x,y,t) = M(y,x,t)$;

**e.** $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$;

**f.** $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;

**g.** $N(x,y,t) \ge 0$;

**h.** $N(x,y,t) = 0$ if and only if $x = y$;

**i.** $N(x,y,t) = N(y,x,t)$;

**j.** $N(x,y,t) \Diamond N(y,z,s) \ge N(x,z,t+s)$;

**k.** $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then $(M,N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and degree of non-nearness between $x$ and $y$ with respect to $t$ respectively [11].

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**TEMPORAL FUZZY NEGATION, T-NORM AND T-CONORM**

In this section the concepts of temporal fuzzy $t-$ norm, temporal fuzzy $t-$ conorm and temporal fuzzy negation, which will be used in defining the concept of temporal intuitionistic fuzzy metric spaces, are defined by adding the time parameter as follows.

**Definition 10.** Let $T$ be a time set. If the mapping $*_t:\left(\left[0,1\right]\times\left[0,1\right]\right)\times T\rightarrow\left[0,1\right]$ is satisfied following the condition for a fixed time moment $t\in T$, it is called temporal fuzzy triangular norm (temporal $t-$norm) at time moment $t$:

1. $*_t\left(\left(x,1\right),t\right)=x$ (bounded condition);
2. $*_t\left(\left(x,y\right),t\right)=*_t\left(\left(y,x\right),t\right)$ (commutativity);
3. $*_t\left(\left(x,*_t\left(\left(y,z\right),t\right)\right),t\right)=*_t\left(\left(*_t\left(\left(x,y\right),t\right),\mathrm{z}\right),t\right)$ (associativity);
4. $*_t\left(\left(a,b\right),t\right)\leq *_t\left(\left(c,d\right),t\right)$ if $a\leq c$ , $b\leq d$ (monotonicity).

**Definition 11.** Let $T$ be a time set. If the mapping $\Diamond_t:\left(\left[0,1\right]\times\left[0,1\right]\right)\times T\rightarrow\left[0,1\right]$ is satisfied following the condition for a fixed time moment $t\in T$, $\Diamond_t$ is called temporal intuitionistic fuzzy triangular conorm (temporal $s-$ norm) at time moment $t$:

1. $\Diamond_t\left(\left(x,0\right),t\right)=x$ (bounded condition);
2. $\Diamond_t\left(\left(x,y\right),t\right)=\Diamond_t\left(\left(y,x\right),t\right)$ (commutativity);
3. $\Diamond_t\left(\left(x,\Diamond_t\left(\left(y,z\right),t\right)\right),t\right)=\Diamond_t\left(\left(\Diamond_t\left(\left(x,y\right),t\right),z\right),t\right)$ (associativity);
4. $\Diamond_t\left(\left(a,b\right),t\right)\leq\Diamond_t\left(\left(c,d\right),t\right)$ if $a\leq c$, $b\leq d$ (monotonicity).

Unlike the t-norm and t-conorm defined in the fuzzy set theory, these temporal fuzzy t-norm and t-conorm gain variable and adaptive structure with the time parameter. This provides the opportunity for conjunctions and disconjunctions represented by t-norms and t-conorms to change over time. In addition, the use of time-dependent and instantaneous results to obtain a general judgment can be achieved by aggregation operators.

**Example 1.** Let $X$ be a non-empty set, $T$ be a time set and $\alpha:T\rightarrow R-\{1\}$ be a continuous function. Define for all $x,y\in\left[0,1\right]$ , $t\in T$ $\Delta_t:\left(\left[0,1\right]\times\left[0,1\right]\right)\times T\rightarrow\left[0,1\right]$ and $\nabla_t:\left(\left[0,1\right]\times\left[0,1\right]\right)\times T\rightarrow\left[0,1\right]$ such that

$$\Delta_t\left(\left(x,y\right),t\right)=\begin{cases}xy & , \quad y=1\\ \alpha\left(t\right)xy & , \quad y\neq 1\end{cases}$$

and

$$\nabla_t\left(\left(x,y\right),t\right)=\begin{cases}x+y-xy & , \quad x=0 \text{ or } y=0\\ x+y-\alpha\left(t\right)xy, & \text{otherwise.}\end{cases}$$

Then it is easily seen that $\Delta_t$ and $\nabla_t$ are temporal fuzzy $t-$norm and $t-$conorm on $X$ and $T$ respectively. Clearly, in the case of $\alpha\left(t\right)=1$, $\Delta_t$ and $\nabla_t$ transform into the ordinary product and probabilistic sum.

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**Example 2.** Let $X$ be a non-empty set, $T$ be a time set and $\alpha : T \to R - \{1\}$ be a continuous function. Define for all $x, y \in [0,1]$, $t \in T$ $\lhd_t : ([0,1]\times[0,1])\times T \to [0,1]$ and $\rhd_t : ([0,1]\times[0,1])\times T \to [0,1]$ such that

$$\lhd_t((x,y),t) = \log_{\alpha(t)}\left(1 + \frac{\left(\alpha(t)^x - x\right)\left(\alpha(t)^y - y\right)}{\alpha(t)-1}\right)$$

and

$$\rhd_t((x,y),t) = 1 - \lhd_t^2(((1-x),(1-y)),t).$$

Then it is easily seen that $\lhd_t$ and $\rhd_t$ are temporal fuzzy $t-$norm and $t-$conorm on $X$ and $T$.

Another important operation in the fuzzy set theory is negation, which is the generalisation of the complementation in classical set theory. The temporal generalisation of fuzzy negation is defined as follows.

**Definition 12.** Let $T$ be a time set. If $N_t : [0,1]\times T \to [0,1]$ satisfies the following conditions, $N_t$ is called temporal fuzzy negation at fixed time moment $t \in T$:

i. $N_t$ is non-increasing mapping for the first variable and continuous for the second variable;
ii. $N_t(0,t) = 1$ and $N_t(1,t) = 0$ at fixed time moment $t \in T$.

If a temporal fuzzy negation is monotonously decreasing at the first component $(x < y \Rightarrow N(x,t) > N(y,t))$ and continuous at both terms, it is called a strict negation. If a strict negation $N$ is an involution ($N(N(x)) = x$ for all $x \in X$), it is called a strong negation.

**Example 3.** The function $N_t : [0,1]\times T \to [0,1]$, which is defined by $N_t(x,t) = \dfrac{1-x}{1+\lambda(t)x}$ where $\lambda : T \to (-1,\infty)$, is a temporal fuzzy strong negation. In the case $\lambda(t) = 0$ this negation becomes the fuzzy standard negation.

**Example 4.** The function $N_t : [0,1]\times T \to [0,1]$, which is defined by $N_t(x,t) = \sqrt[\alpha(t)]{1 - x^{\alpha(t)}}$ where $\alpha : T \to (0,\infty)$, is a temporal fuzzy strong negation. In the case $\alpha(t) = 1$ this negation becomes the fuzzy standard negation.

**Definition 13.** Let $*_t$ be a temporal intuitionistic fuzzy $t-$norm, $\Diamond_t$ be a temporal intuitionistic fuzzy $t-$conorm and $N_t$ be a temporal intuitionistic fuzzy strong negation. If the ordered triplet $(\Diamond_s, *_t, N_t)$ satisfies De Morgan's condition:

$$\Diamond_t((x,y),t) = N_t\left(*_t\left((N_t(x,t), N_t(y,t)),t\right),t\right),$$

the triplet is called temporal intuitionistic fuzzy De Morgan triplet.

This definition, which is the generalisation of De Morgan's Rule in classical set theory, has an important role in the consistency of systems established with fuzzy set and its generalisations.

**Proposition 1.** If $(\Diamond_s, *_t, N_t)$ is a temporal fuzzy De Morgan triplet at time moment $t$, then the equation $*_t((x,y),t) = N_t\left(\left(\Diamond_t\left(N_t(x,t), N_t(y,t)\right),t\right),t\right)$ is satisfied.

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**Proof.** Since $(\Diamond_s, *_t, N_t)$ at the time moment $t$ is a temporal fuzzy De Morgan triplet, the equation $\Diamond_t\left(\left(N_t(x,t), N_t(y,t)\right),t\right) = N_t\left(*_t\left(\left(N_t\left(N_t(x,t),t\right), N_t\left(N_t(y,t),t\right)\right),t\right),t\right)$ can be written. Here, $\Diamond_t\left(\left(N_t(x,t), N_t(y,t),t\right)\right) = N_t\left(*_t\left((x,t),(y,t)\right)\right)$ is obtained. If both sides of the last equation are subjected to $N_t$ negation, $*_t\left((x,t),\mathrm{t}\right) = N_t\left(\left(\Diamond_t\left(N_t(x,t), N_t(y,t)\right),t\right)\right)$ equality is obtained because $N_t$ is the temporal fuzzy strong negation.

**Example 5.** $(\Delta_t, \nabla_t, N_t)$, where $N_t(x,t) = \dfrac{1-x}{1+\lambda(t)x}$ and $\lambda : T \to (-1,\infty)$, forms a temporal fuzzy De Morgan triplet.

## TEMPORAL INTUITIONISTIC FUZZY METRIC SPACES

As it is known, the membership and non-membership degrees of temporal intuitionistic fuzzy sets change with time. With a similar approach, the concept of temporal intuitionistic fuzzy metric spaces has emerged with this new approach, which is based on the idea that the degrees of nearness and non-nearness, defined in the concept of intuitionistic fuzzy metric spaces, change with time. This definition, which is based on the idea of dynamising the intuitionistic fuzzy metric concept defined by Park [11] by adding the time parameter, is given below:

**Definition 14.** A 6-tuple $(X, T, M_t, N_t, *_t, \Diamond_t)$ is said to be a temporal intuitionistic fuzzy metric space if $X$ is an arbitrary non-empty set and $T$ is the time set, $*_t$ is a temporal continuous $t-$ norm, $\Diamond_t$ is a temporal continuous $t-$ conorm, $M_t$ and $N_t$ are functions on $X \times X \times (0,\infty) \times T$ to $[0,1]$, satisfying the following conditions for fixed $t \in T$ and all $x, y, z \in X$ and $n, n_1, n_2 > 0$:

i. $M_t(x,y,n,t) + N_t(x,y,n,t) \leq 1$;
ii. $M_t(x,y,n,t) \geq 0$;
iii. $M_t(x,y,n,t) = 1$ if and only if $x = y$;
iv. $M_t(x,y,n,t) = M_t(y,x,n,t)$;
v. $*_t\left(\left(M_t(x,y,n_1,t), M_t(y,z,n_2,t)\right),t\right) \leq M_t(x,z,n_1+n_2,t)$;
vi. $M_t(x,y,.,t) : (0,\infty) \to (0,1]$ is continuous;
vii. $N_t(x,y,n,t) \leq 1$
viii. $N_t(x,y,n,t) = 0$ if and only if $x = y$;
ix. $N_t(x,y,n,t) = N_t(y,x,n,t)$
x. $\Diamond_t\left(\left(N_t(x,y,n_1,t), N_t(y,z,n_2,t)\right),t\right) \geq N_t(x,z,n_1+n_2,t)$
xi. $N_t(x,y,.,t) : (0,\infty) \to (0,1]$ is continuous;
xii. $M_t(x,y,n,.) : T \to (0,1]$ and $N_t(x,y,n,.) : T \to (0,1]$ are continuous.

Then $(M_t, N_t)$ is called a temporal intuitionistic fuzzy metric on X. The functions $M_t(x,y,n,t)$ and $N_t(x,y,n,t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ at time moment $t$ respectively.

**Proposition 2.** Let $(X,d)$ be a classical metric space, $T$ be a time set and $\alpha : T \to R^+$ be a function. Denote $*_t\left((x,y),t\right) = \min\{x,y\}$ and $\Diamond_t\left((x,y),t\right) = \max\{x,y\}$ and let

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$M_t : X \times X \times (0,\infty) \times T \to [0,1]$ and $N_t : X \times X \times (0,\infty) \times T \to [0,1]$ for each $x, y \in X$ and $n, n_1, n_2 > 0$ be defined as follows:

$$M_t\left((x,y),n,t\right) = \frac{n^{\alpha(t)}}{n^{\alpha(t)} + d(x,y)}$$

and

$$N_t\left((x,y),n,t\right) = \frac{d(x,y)}{n^{\alpha(t)} + d(x,y)}.$$

Then the ordered 6-tuple $(X,T,M_t,N_t,*_t,\Diamond_t)$ is a temporal intuitionistic fuzzy metric space.

**Proposition 3.** Let $M_t$: $\mathbb{N} \times \mathbb{N} \times (0,\infty) \times T \to [0,1]$ and $N_t$: $\mathbb{N} \times \mathbb{N} \times (0,\infty) \times T \to [0,1]$ be defined as follows:

$$M_t(x,y,n,t) = \begin{cases} \left(\dfrac{x}{y}\right)^{\beta(t)}, & x \le y \\ \left(\dfrac{y}{x}\right)^{\beta(t)}, & y \le x \end{cases}$$

$$N_t(x,y,n,t) = \begin{cases} \left(\dfrac{y-x}{y}\right)^{\beta(t)}, & x \le y \\ \left(\dfrac{x-y}{x}\right)^{\beta(t)}, & y \le x \end{cases}.$$

where $\beta : T \to \mathbb{N}$ is a non-decreasing function. Then the ordered 6-tuple $(X,T,M_t,N_t,\Delta_t,\nabla_t)$ constitutes a temporal intuitionistic fuzzy metric space with these assumptions.

## TOPOLOGICAL SPACE GENERATED BY TEMPORAL INTUITIONISTIC FUZZY METRIC SPACE AND BASIC PROPERTIES

**Definition 15.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be a temporal intuitionistic fuzzy metric space, $r \in (0,1)$, $n > 0$ and $t \in T$. The set $B_t(x,r,n,t) = \{y \in X; M_t(x,y,n,t) > 1-r, N_t(x,y,n,t) < r\}$ is called the temporal open ball with centre $x$ and radius $r$ at time $t$.

The difference between open balls generated by temporal intuitionistic fuzzy metric space and open balls generated by intuitionistic fuzzy metric space is that the former change with time. Basically, this definition, which is equivalent to that of the open ball defined in classical, fuzzy and intuitionistic fuzzy metric spaces, is a broader definition than that of the previous spaces and also preserves the properties of open balls in these metric spaces. The most important ones among these properties are given by the following theorems.

**Theorem 1.** Every temporal open ball $B_t(x,r,n,t)$ is an open set.

**Proof.** Straightforward

From the above theorem, the temporal open balls obtained by the temporal intuitionistic fuzzy metric are open sets. Thus, it is understood that a temporal topology can be produced with a temporal intuitionistic fuzzy metric space.

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**Theorem 2.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be a temporal intuitionistic metric space over the time set $T$ and non-empty set $X$. Then

$$\tau_{(M_t,N_t)} = \{A \subset X : \forall x \in A \text{ için } \exists t > 0 \text{ ve } r \in (0,1) \text{ such that } B_t(x,r,n,t) \subset A\}$$

defined as $\tau_{(M_t,\mathrm{N}_t)}$ is a topology on $X$ at time moment $t$.

**Proof.** $(T_1)$ Since $\varnothing$ set has no elements, the temporal open ball that accepts the elements of $\varnothing$ as centre is also empty. In that case $\varnothing \in \tau_{(M_t,\mathrm{N}_t)}$. Since $B_{(M_t,N_t)}(x,r,n,t) \subset X$ becomes $X \in \tau_{(M_t,\mathrm{N}_t)}$ for each $\forall x \in X$ and $r > 0$, so $X$ becomes $\varnothing \in \tau_{(M_t,\mathrm{N}_t)}$.

$(T_2)$ Let $A_1(t), A_2(t), A_3(t), \ldots, A_n(t) \in \tau_{(M_t,\mathrm{N}_t)}$.

(i) $\bigcap_{i=1}^{n} A_i(t) = \varnothing \Rightarrow \bigcap_{i=1}^{n} A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$.

(ii) Let $\bigcap_{i=1}^{n} A_i(t) \neq \varnothing$ and $x \in \bigcap_{i=1}^{n} A_i(t)$. In this case since $x \in A_i(t)$ and $A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$ for $\forall i = 1,2,\ldots,n$, then $0 < r_i < 1$ and $t_i > 0$, so that $B_{(M_t,N_t)}(x,r_i,\mathrm{n},t_i) \subset A_i(t)$ for $\forall i \in \{1,2,\ldots,n\}$. Let $r = \min\{r_i : i = 1,\ldots,n\}$ and $t = \max\{t_i : i = 1,\ldots,n\}$. Therefore, $0 < r < 1$ and $t > 0$ such that $B_{(M_t,N_t)}(x,r,\mathrm{n},t) \subset B_{(M_t,N_t)}(x,r_i,\mathrm{n},t_i) \subset A_i(t)$ for $\forall_i = 1,2,\ldots,n$. In this case $x \in B_{(M_t,N_t)}(x,r,\mathrm{n},t) \subset \bigcap_{i=1}^{n} A_i(t)$ is obtained, which means $\bigcap_{i=1}^{n} A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$.

$(T_3)$ Let $\bigcup_{i \in I} A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$ when $\{A_i(t)\}_{i \in I} \subset \tau_{(M_t,\mathrm{N}_t)}$. If $\bigcup_{i \in I} A_i(t) = \varnothing$, it is clear that $\bigcup_{i \in I} A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$.

Let $x \in \bigcup_{i \in I} A_i(t) \neq \varnothing$. In this case $x \in A_{i_0}(t)$ for $\exists i_0 \in I$. Since $A_{i_0}(t) \in \tau_{(M_t,\mathrm{N}_t)}$, then $0 < r < 1$ and $t > 0$, so that $B_{(M_t,N_t)}(x,r,\mathrm{n},t) \subset A_{i_0}(t)$. Therefore $\bigcup_{i \in I} A_i(t) \in \tau_{(M_t,\mathrm{N}_t)}$.

Thus, $\tau_{(M_t,\mathrm{N}_t)}$ is a topology over $X$.

Since the open ball of the topology generated by the $(X,T,M_t,N_t,*_t,\Diamond_t)$ temporal intuitionistic fuzzy metric space will change with time, the topology itself will change with time, hence the name temporal topology. Since this is a classical topology, many properties provided by classical metric spaces can also be provided by temporal topological spaces. This claim is supported by the theorems and proof methods in several studies [e.g. 2, 20-25]. In this manner it is aimed to obtain a new and more general metric topology in which the basic topological properties of fuzzy and intuitionistic fuzzy metric spaces are preserved. In the following the fundamental properties of classical, fuzzy and intuitionistic fuzzy metric spaces are generalised in the temporal fuzzy metric space. Many proofs are similar to their previous counterparts.

**Theorem 3.** The temporal intuitionistic fuzzy metric space $(X,T,M_t,N_t,*_t,\Diamond_t)$ is a Hausdorff space.

**Proof.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be a temporal intuitionistic fuzzy metric space and $x, y \in X$, $x \neq y$. To be defined as $r_1 = M_t(x,y,n,t)$, $r_2 = N_t(x,y,n,t)$ and $r = \max\{r_1, 1-r_2\}$, there is $r_3, r_4 \in (0,1)$ such that $*_t((r_3,r_3),t) \geq r_0$ and $\Diamond_t(((1-r_4),(1-r_4)),t) \leq 1-r_0$ for each $r_0 \in (r,1)$ that satisfies the

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condition $r < r_0 < 1$. Let consider open balls $B_t\left(x,1\text{-}r_5,\frac{1}{2}n,t\right)$ and $B_t\left(y,1-r_5,\frac{1}{2}n,t\right)$ to be $r_5 = \max\{r_3,r_4\}$. Obviously, the intersection of the balls is empty, so $B_t\left(x,1-r_5,\frac{1}{2}n,t\right)\cap B_t\left(y,1-r_5,\frac{1}{2}n,t\right)=\varnothing$. Otherwise, if

$$z \in B_t\left(x,1-r_5,\frac{1}{2}n,t\right)\cap B_t\left(y,1-r_5,\frac{1}{2}n,t\right),$$

$$r_1 = M(x,y,n,t) \geq *_t\left(\left(M_t\left(x,z,\frac{1}{2}n,t\right),M_t\left(z,y,\frac{1}{2}n,t\right),t\right)\right) \geq *_t\left((r_5,r_5),t\right)$$

$\geq *_t\left((r_3,r_3),t\right) \geq r_0 > r_1$. On the other hand, the contradictions

$$r_2 = N_t(x,y,n,t) \leq \lozenge_t\left(\left(N_t\left(x,z,\frac{1}{2}n,t\right),N_t\left(z,y,\frac{1}{2}n,t\right)\right),t\right) \leq \lozenge_t\left(\left((1-r_5),(1-r_5)\right),t\right) \leq$$

$$\lozenge_t\left(\left((1-r_4),(1-r_4)\right),t\right) \leq 1-r_0 < r_2$$

are obtained. These contradictions arise from the assumption that $B_t\left(x,1-r_5,\frac{1}{2}n,t\right)\cap B_t\left(y,1-r_5,\frac{1}{2}n,t\right)\neq\varnothing$. Thus, it is understood that the temporal intuitionistic fuzzy metric space $(X,T,M_t,N_t,*_t,\lozenge_t)$ is a Hausdorff space.

The concept of boundedness given in the fuzzy and intuitionistic fuzzy metric spaces is defined for the temporal intuitionistic fuzzy metric spaces as follows.

**Definition 16.** Let $(X,T,M_t,N_t,*_t,\lozenge_t)$ be a temporal intuitionistic fuzzy metric space and $A \subset X$. If there are $t>0$ and $r\in(0,1)$ numbers so that $M_t(x,y,n,t)>1-r$ and $N_t(x,y,n,t)<r$ for each $x,y\in A$, the set $A$ is called temporal intuitionistic fuzzy bounded (TIF-bounded).

**Theorem 4.** Let $(X,T,M_t,N_t,*_t,\lozenge_t)$ be a temporal intuitionistic fuzzy metric space reduced by the $d$ metric and $A$ be a subset of $X$. Subset $A$ is TIF-bounded only if $A$ is bounded.

**Proof.** $(\Rightarrow)$: Let the set $A\subset X$ be TIF-bounded. So $0<r<1$ and $t>0$ such that $M_t(x,y,n,t)>1-r$ for $\forall x,y\in A$. From here, taking $x\in A\subset X$ for $a\in A$, $a\in B_t(x,r,n,t)$ since $M_t(x,a,\mathrm{n},t)>1-r$. This means that $x\in X$, $0<r<1$ and $t>0$ such that $A\subset B_t(x,r,n,t)$. Thus, $A$ is bounded.

$(\Leftarrow)$: Let the set $A\subset X$ be bounded. So $x\in X$, $0<r<1$ and $t>0$ such that $A\subset B_t(x,r,n,t)$. In that case, for $a,b\in A$, $M_t(x,a,n,t)>1-r$ and $M_t(x,\mathrm{b},\mathrm{n},t)>1-r$ since $a,b\in B_t(x,r,n,t)$. From here, $M_t(a,b,n,2t)\geq M_t(a,x,n,t)*M_t(x,b,n,t)>(1-r)*(1-r)$ for $\forall a,b\in A$. On the other hand, for $0<r<1$, there is $0<s<1$ such that $(1-r)*(1-r)>1-s$. If $2t=t'$, we get $t'>0$ and $0<s<1$ such that $M_t(a,b,\mathrm{n},t')>1-s$ for $\forall a,b\in A$. This proves that $A$ is an TIF-bounded set.

**Definition 17.** Let $(X,T,M_t,N_t,*_t,\lozenge_t)$ be a temporal intuitionistic fuzzy metric space, $\tau_{(M_t,N_t)}$ be the topology which is obtained by the fuzzy metric on $X$, and $(x_m)$ be a sequence on $X$. $(x_m)$ is said to converge on *x* with respect to $(X,T,M_t,N_t,*_t,\lozenge_t)$ space at the time moment $t$ if there exists an integer *m* such that $x_m\in B_t(x,r,n,t)$ for all $m\geq m_0$, $0<r<1$ and $n>0$.

**Theorem 5.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be a temporal intuitionistic fuzzy metric space and $\tau_{(M_t,N_t)}$ be the topology on X induced by the fuzzy metric. Then for a sequence $(x_m)$, $x_m \to x$ if and only if $M_t(x_m,x,n,t)\to 1$ and $N_t(x_m,x,n,t)\to 0$ as $m\to\infty$.

**Proof.** Straightforward.

**Definition 18.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be a temporal intuitionistic fuzzy metric space. Then a sequence $\{x_n\}$ in $X$ is said to be Cauchy if for each $\varepsilon>0$ and each $n>0$ there exists an integer $m_0$ such that $M_t(x_m,x_{m'},n,t)>1-\varepsilon$ and $N_t(x_m,x_{m'},n,t)<\varepsilon$ for all $m,m'\geq m_0$ . If $(X,T,M_t,N_t,*_t,\Diamond_t)$ is in a temporal intuitionistic fuzzy metric space at a time moment *t* and each Cauchy sequence is convergent with respect to the $\tau_{(M_t,N_t)}$ topology, then the temporal intuitionistic fuzzy metric space $(X,T,M_t,N_t,*_t,\Diamond_t)$ is complete at time moment *t*.

**Theorem 6.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be an intuitionistic fuzzy metric space such that every Cauchy sequence in $X$ has a convergent subsequence. Then $(X,T,M_t,N_t,*_t,\Diamond_t)$ is complete.

**Proof.** Let $\{x_m\}$ be a Cauchy sequence and let $\{x_{m'}\}$ be a subsequence of $\{x_m\}$ that converges to $x$. We prove that $x_m\to x$ . Let $n>0$ and $\varepsilon>0$. Choose $r\in(0,1)$ such that $(1-r)*(1-r)\geq 1-\varepsilon$ and $r\Diamond r\leq\varepsilon$ . Since $\{x_m\}$ is a Cauchy sequence, there is an integer $m_0$ such that $M_t\left(x_m,x,\frac{n}{2},t\right)>1-r$ and $N_t\left(x_m,x,\frac{n}{2},t\right)<r$ for all $m,m'\geq m_0$. Since $x_{m'}\to x$, there is a positive integer $m^{'}$ such that $m^{'}>m_0$, $M_t\left(x_{m^{'}},x,\frac{n}{2},t\right)>1-r$ and $N_t\left(x_{m^{'}},x,\frac{n}{2},t\right)<r$. Then if $m\geq m_0$,

$$M_t(x_m,x_{m'},n,t)\geq *_t\left(\left(M_t\left(x_m,x,\frac{n}{2},t\right),M_t\left(x_{m'},x,\frac{n}{2},t\right)\right),t\right)>(1-r)*(1-r)\geq 1-\varepsilon$$

and

$$N_t(x_m,x_{m'},n,t)\leq \Diamond_t\left(\left(N_t\left(x_m,x,\frac{n}{2},t\right),N_t\left(x_{m'},x,\frac{n}{2},t\right)\right),t\right)<r\Diamond r\leq\varepsilon .$$

Therefore, $x_m\to x$ and hence $(X,T,M_t,N_t,*_t,\Diamond_t)$ is complete.

Another fundamental property of the classical (also fuzzy and intuitionistic fuzzy) separable metric spaces is second countability. This property is also preserved in temporal intuitionistic fuzzy metric spaces.

**Theorem 7.** Every separable temporal intuitionistic fuzzy metric space is second countable.

**Proof.** Let $(X,T,M_t,N_t,*_t,\Diamond_t)$ be the given separable temporal intuitionistic fuzzy metric space. Let $A(t)=\{x_n : n\in\mathbb{N}\}$ be a countable dense subset of $X$ . Consider the family $\beta=\left\{B_{(M_t,N_t)}\left(x_j,\frac{1}{k},\frac{1}{k},t\right): j,k\in\mathbb{N}\right\}$. Then $\beta$ is countable. We claim that $\beta$ is a base for the family of all open sets in $X$ . Let $U(t)$ be any open set in $X$ and let $x\in U(t)$. Then there exist $t\in T$ and $r\in(0,1)$ such that $B_{(M_t,N_t)}(x,r,n,\mathrm{t})\subset U(t)$ . Since $r\in(0,1)$, we can choose an $s\in(0,1)$ such that

$*_t\left(\left(\left(1-s\right)\left(1-s\right)\right),t\right)>1-r$ and $\Diamond_t\left(\left(s,s\right),t\right)<r$. Take $m\in\mathbb{R}$ such that $\frac{1}{m}<\min\left\{s,\frac{n}{2}\right\}$. Since $A$ is dense in $X$ , there exists $x_j\in A$ such that $x_j\in B_t\left(x,\frac{1}{m},\frac{1}{m},t\right)$. Now if $y\in B_t\left(x_j,\frac{1}{m},\frac{1}{m},t\right)$, then

$$M\left(x,y,n,t\right)\geq *_t\left(\left(M_t\left(x,x_j,\frac{n}{2},t\right),M_t\left(y,x_j,\frac{n}{2},t\right)\right),t\right)\geq M_t\left(x,x_j,\frac{1}{m},t\right)*M_t\left(y,x_j,\frac{1}{m},t\right)$$

$$\geq *_t\left(\left(\left(1-\frac{1}{m}\right),\left(1-\frac{1}{m}\right)\right),t\right)\geq *_t\left(\left(\left(1-s\right),\left(1-s\right)\right),t\right)>1-r$$

and

$$N_t\left(x,y,n,t\right)\leq N_t\left(x,x_j,\frac{n}{2},t\right)\Diamond N_t\left(y,x_j,\frac{n}{2},t\right)\leq N_t\left(x,x_j,\frac{1}{m},\mathrm{t}\right)\Diamond N_t\left(y,x_j,\frac{1}{m},t\right)$$

$$\leq \Diamond_t\left(\left(\frac{1}{m},\frac{1}{m}\right),t\right)\leq \Diamond_t\left(\left(s,s\right),t\right)<r.$$

Thus, $y\in B_t\left(x,r,n,t\right)\subset U\left(t\right)$ and hence $\beta$ is a base.

## CONCLUSIONS

In this study an adaptive metric suitable for the structure of spatio-temporal variables has been defined. It has been shown that this defined metric space preserves the fundamental structure of previous metric spaces. In addition, temporal t-norm, temporal t-conorm and temporal negation obtained by adding the time parameter can be used in many different systems. In future we will work on innovations this new metric brings to engineering problems and especially to deep learning.

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