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Full Paper

μ-Proximity structure via hereditary classes

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Abstract: A new generalised μ-proximity structure is obtained using hereditary class on a set.

Keywords: generalised topology, hereditary class, μ-proximity

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INTRODUCTION

Császár [1-3] introduced and investigated the notions of generalised topology and hereditary class. Then many authors such as Carpintero et al. [4], Renukadevi and Vimaladevi [5] and Qahis and Noiri [6] have used these concepts to extend classical topological concepts.

Efremovič [7] introduced proximity structure, which plays an important role in many problems of topological spaces such as compactification. Then Lodato and others [8-11] investigated generalised proximity structures. Especially, Hosny and Tantawy [8] constructed a new proximity structure via ideals and Mukherjee et al. [10] defined μ-proximity and proved that every proximity space is a μ-proximity space. In addition, they introduced quasi μ-proximity as a generalisation of μ-proximity.

In this paper we construct a kind of μ-proximity via hereditary classes. Firstly, we define a local function with respect to μ-proximity and hereditary classes and give its basic properties. Then by using this function, we establish a new quasi μ-proximity and investigate its relations to μ-proximity.

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PRELIMINARIES

Let X be a non-empty set and let $\wp(X)$ denote the power set of X. Then $\mu \subset \wp(X)$ is called a generalised topology (GT) on X [1, 2] if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ imply $G = \bigcup_{i\in I} G_i \in \mu$. The pair (X, μ) is called a generalised topological space (GTS). The elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. Let (X, μ) be a GTS and $A \subset X$. The μ-closure of A, denoted by $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the μ-interior of A, denoted by $i_\mu(A)$, is the union of all μ-open sets contained in A. Then $c_\mu(A)$ is a μ-closed set and $x \in c_\mu(A)$ if and only if $x \in G \in \mu$ implies $G \cap A \neq \emptyset$. A map $\varphi: X \longrightarrow \wp(\wp(X))$ is called a generalised neighbourhood system [2] on X if for each $x \in X$, $V \in \varphi(x)$ implies $x \in V$. Then $V \in \varphi(x)$ is called a generalised neighbourhood of $x \in X$. If μ is a GT on X, then we can define a generalised neighbourhood system φ_μ on X by $\varphi(x) = \{A: x \in G \subset A \; for\ some\ G \in \mu\}$ for $x \in X$. In addition, μ is normal [12] if and only if whenever F and F' are μ-closed sets such that $F \cap F' = \emptyset$, there exist μ-open sets G and G'satisfying $F \subset G, F' \subset G'$ and $G \cap G' = \emptyset$.

A non-empty family $\mathfrak{H}$ of subsets of a non-empty set X is called a hereditary class [3] if $A \subset B$ and $B \in \mathfrak{H}$; then $A \in \mathfrak{H}$. $\mathfrak{H}$ is said to be μ-codense [3] if $\mu \cap \mathfrak{H} = \{\emptyset\}$ and strongly μ-codense [3] if $G, G' \in \mu$ and $G \cap G' \in \mathfrak{H}$; then $G \cap G' = \emptyset$.

A binary relation δ_μ on $\wp(X)$ is called a μ-proximity [10] on X if δ_μ satisfies the following conditions for $A, B, C, D \in \wp(X)$:

(1) $A\delta_\mu B$ if and only if $B\delta_\mu A$;

(2) If $A\delta_\mu B$, $A \subset C$ and $B \subset D$, then $C\delta_\mu D$;

(3) $x\delta_\mu x$ for each $x \in X$;

(4) $A\overline{\delta_\mu}B$ implies there exists C such that $A\overline{\delta_\mu}C$ and $(X\backslash C)\overline{\delta_\mu}B$.

Also, δ_μ is said to be quasi μ-proximity if it satisfies (2), (3) and (4).

Proposition 1 [10]. Let (X, δ_μ) be a μ-proximity space (or quasi μ-proximity space) and let a subset A of X be defined to be δ_μ-closed if and only if $x\delta_\mu A$ implies $x \in A$. Then the collection of complements of all δ_μ-closed sets produce a GT $\mu = \tau(\delta_\mu)$ on X.

Proposition 2 [10]. Let (X, δ_μ) be a μ-proximity space (or quasi μ-proximity space) and $\mu = \tau(\delta_\mu)$. Then $c_{\delta_\mu}(A) = \{x: x\delta_\mu A\}$ is the μ-closure of $A \subset X$. Also, $G \in \tau(\delta_\mu)$ if and only if $x\overline{\delta_\mu}(X\backslash G)$ for each $x \in G$.

In this paper the members of $\tau(\delta_\mu)$ will be called δ_μ-open sets.

Lemma 1 [10]. Let (X, δ_μ) be a μ- proximity space and $A, B \subset X$. Then

$$A\delta_\mu B \;\Leftrightarrow c_{\delta_\mu}(A)\, \delta_\mu\, c_{\delta_\mu}(B)$$

where the μ-closure is taken with respect to $\tau(\delta_\mu)$.

Definition 1 [10]. Let (X, δ_μ) be a μ-proximity space and $A, B \subset X$. B is called a δ_μ-neighbourhood of A if $A\overline{\delta_\mu}(X\backslash B)$; it is denoted by $A \ll_\mu B$.

Theorem 1 [10]. Let (X, δ_μ) be a μ-proximity space. Then the relation $\ll_\mu$ satisfies the following conditions for $A, B, C, D \in \wp(X)$:

(1) $A \ll_\mu B$ implies $A \subset B$;

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(2) $A \subset B \ll_{\mu} C \subset D$ implies $A \ll_{\mu} D$;
(3) $A \ll_{\mu} B$ implies $(X \backslash B) \ll_{\mu} (X \backslash A)$;
(4) $A \ll_{\mu} B$ implies there exists $C \subset X$ such that $A \ll_{\mu} C \ll_{\mu} B$.

Remark 1 [10]. Let (X, δ_{μ}) be a μ-proximity space and $A \subset X$. Then each δ_{μ}-neighbourhood is also a $\tau(\delta_{\mu})$-neighbourhood of A.

HEREDITARY μ-PROXIMITY SPACES

In this section we define the local function with respect to a hereditary class and μ-proximity. Also, we study its several properties.

Definition 2. A μ-proximity space (X, δ_{μ}) with a hereditary class $\mathfrak{H}$ is hereditary μ-proximity space denoted by $(X, \delta_{\mu}, \mathfrak{H})$. For each subset A of X is defined the local function of A with respect to δ_{μ} and $\mathfrak{H}$ as follows:

$$A^*(\delta_{\mu}, \mathfrak{H}) = \cup\{x \in X: U \cap A \notin \mathfrak{H} \; for \; all \; \delta_{\mu} - neighbourhood \; U \; of \; x\}$$

We will simply write A^* or $A^*(\mathfrak{H})$ for $A^*(\delta_{\mu}, \mathfrak{H})$.

Proposition 3. Let $(X, \delta_{\mu}, \mathfrak{H})$ be a hereditary μ-proximity space and $A, B \subset X$. Then the following hold:
(1) $A \subset B$ implies $A^* \subset B^*$;
(2) $A \in \mathfrak{H}$ implies $A^* = \emptyset$.

Proof.
(1) Let $x \in A^*$ for $x \in X$. We have $U \cap A \notin \mathfrak{H}$ for each δ_{μ}-neighbourhood U of x. This implies $U \cap B \notin \mathfrak{H}$ since $A \subset B$. Then we have $x \in B^*$.
(2) Since $A \in \mathfrak{H}$, it is obvious by Definition 2. □

In the following example it is shown that A^* and A are independent of each other and the local function with respect to δ_{μ} and $\mathfrak{H}$ is not closed under finite union.

Example 1. Let $\mathfrak{H}$ be a hereditary class and let δ be an indiscrete proximity on any set X. That is, $A\delta B$ for every pair of non-empty subsets A and B of X. Since δ is a proximity, then it is also a μ-proximity. If $A \in \mathfrak{H}$, then $A^* = \emptyset$ and if $A \notin \mathfrak{H}$, then $A^* = X$. Also, let $A, B \in \mathfrak{H}$ and $A \cup B \notin \mathfrak{H}$. Then we have $(A \cup B)^* = X \neq A^* \cup B^* = \emptyset$.

Theorem 2. Let $(X, \delta_{\mu}, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then $A^*(\tau(\delta_{\mu}), \mathfrak{H}) \subset A^*(\delta_{\mu}, \mathfrak{H})$.

Proof. Let $x \notin A^*(\delta_{\mu}, \mathfrak{H})$. Then there exists a δ_{μ}-neighbourhood U of x such that $U \cap A \in \mathfrak{H}$. Since U is a δ_{μ}-neighbourhood of x, it is also $\tau(\delta_{\mu})$-neighbourhood of x. Thus, we get $x \notin A^*(\tau(\delta_{\mu}), \mathfrak{H})$.
□

Proposition 4. Let (X, δ_{μ}) be a μ-proximity space and let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two hereditary classes on X. $\mathfrak{H}_1 \subset \mathfrak{H}_2$ implies $A^*(\mathfrak{H}_2) \subset A^*(\mathfrak{H}_1)$ for $A \subset X$.

Proof. Let $x \in A^*(\mathfrak{H}_2)$. Then for every δ_{μ}-neighbourhood U of x, $U \cap A \notin \mathfrak{H}_2$. This implies that $U \cap A \notin \mathfrak{H}_1$ by the hypothesis. Thus, we obtain $x \in A^*(\mathfrak{H}_1)$. □

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Definition 3. Let δ_μ^1 and δ_μ^2 be two μ-proximities on a non-empty set X. Then δ_μ^2 is called finer than δ_μ^1, denoted by $\delta_\mu^1 < \delta_\mu^2$, if $A\delta_\mu^2 B$ implies $A\delta_\mu^1 B$ for $A, B \subset X$.

Proposition 5. Let δ_μ^1 and δ_μ^2 be two μ-proximities on a non-empty set X. If $\delta_\mu^1 < \delta_\mu^2$, then $\tau(\delta_\mu^1) \subset \tau(\delta_\mu^2)$.

Proof. Let $A \in \tau(\delta_\mu^1)$. Then $x\, \overline{\delta_\mu^1}(X\setminus A)$ for each $x \in A$. Since $\delta_\mu^1 < \delta_\mu^2$, it follows that $x\, \overline{\delta_\mu^2}(X\setminus A)$. Thus, we get $A \in \tau(\delta_\mu^2)$. □

Proposition 6. Let δ_μ^1, δ_μ^2 be two μ-proximities on a non-empty set X and let δ_μ^2 be finer than δ_μ^1. Then for any hereditary class $\mathfrak{H}$ on X and for $A \subset X$, we have $A^*(\delta_\mu^2, \mathfrak{H}) \subset A^*(\delta_\mu^1, \mathfrak{H})$.

Proof. Let $x \notin A^*(\delta_\mu^1, \mathfrak{H})$. Then there exists a δ_μ^1-neighbourhood U of x such that $U \cap A \in \mathfrak{H}$. By hypothesis, U is also δ_μ^2-neighbourhood of x. Thus, we obtain $x \notin A^*(\delta_\mu^2, \mathfrak{H})$. □

Lemma 2. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$.

(1) If $\mathfrak{H} = \{\emptyset\}$, then $A^*(\delta_\mu, \{\emptyset\}) = c_{\delta_\mu}(A)$.
(2) If $\mathfrak{H} = \wp(X)$, then $A^*(\delta_\mu, \wp(X)) = \emptyset$.

Proof.

(1) Let $x \notin c_{\delta_\mu}(A)$. Then we have $x\, \overline{\delta_\mu} A$ by Proposition 2. This implies that $X\setminus A$ is a δ_μ-neighbourhood of x. Since $A \cap (X\setminus A) = \emptyset \in \mathfrak{H}$, we have $x \notin A^*(\delta_\mu, \{\emptyset\})$. For the other inclusion, let $x \notin A^*(\delta_\mu, \{\emptyset\})$. Then there exists a δ_μ-neighbourhood U of x such that $U \cap A \in \mathfrak{H} = \{\emptyset\}$. This implies $U \cap A = \emptyset$. So we have $x\, \overline{\delta_\mu} A$. Thus, we get $x \notin c_{\delta_\mu}(A)$.
(2) For each $x \in X$ and for each δ_μ-neighbourhood U of x, $U \cap A \in \mathfrak{H} = \wp(X)$. So $A^*(\delta_\mu, \wp(X)) = \emptyset$. □

Proposition 7. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then the following hold.

(1) $A^* \subset c_{\delta_\mu}(A)$.
(2) $x\, \delta_\mu A^*$ implies $x\delta_\mu A$.

Proof.

(1) Let $x \notin c_{\delta_\mu}(A)$. Then we have $x\overline{\delta_\mu}A$, which implies that $x \ll_\mu X\setminus A$. So $X\setminus A$ is a δ_μ-neighbourhood of x such that $A \cap (X\setminus A) = \emptyset \in \mathfrak{H}$. Thus, we obtain $x \notin A^*$.
(2) Let $x\delta_\mu A^*$. Then we obtain $x\, \delta_\mu c_{\delta_\mu}(A)$ from (1). Since $c_{\delta_\mu}(A)$ is δ_μ-closed, we have $x \in c_{\delta_\mu}(A)$. Thus, we get $x\delta_\mu A$. □

The following example shows that the converse implication of Proposition 7(2) may not be true in general.

Example 2. Let $\mathfrak{H} = \{\emptyset, \{b\}, \{c\}\}$ be a hereditary class and δ be a discrete proximity on $X = \{a, b, c\}$. That is, $A\delta B$ if and only if $A \cap B \neq \emptyset$ for $A, B \subset X$. δ is also a μ-proximity on X. Let $A = \{a, c\} \subset X$. Then $c\delta A$, but $c\, \bar{\delta} A^* = \{a\}$.

Lemma 3. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. If G is a δ_μ-open set and $G \cap A \in \mathfrak{H}$, then $G \cap A^* = \emptyset$.

Proof. Let G be a δ_μ-open set and assume to the contrary that $G \cap A^* \neq \emptyset$. Then there exists $x \in X$ such that $x \in G$ and $x \in A^*$. Since G is a δ_μ-open set containing x, we get $x \ll_\mu G$. Therefore, we have $G \cap A \notin \mathfrak{H}$. □

Proposition 8. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then A^* is δ_μ-closed.

Proof. Let $x \notin A^*$. Then there exists a δ_μ-neighbourhood V of x such that $V \cap A \in \mathfrak{H}$. Since V is a δ_μ-neighbourhood of x, it is also a $\tau(\delta_\mu)$-neighbourhood of x. This implies that there exists a δ_μ-open set G containing x such that $G \subset V$. So we have $G \cap A \in \mathfrak{H}$. By Lemma 3, we obtain $G \cap A^* = \emptyset$. Thus, we get $x\overline{\delta_\mu}A^*$. Hence A^* is δ_μ-closed. □

Proposition 9. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then $A \subset A^*$ if and only if $A^* = c_{\delta_\mu}(A)$.

Proof. Necessity: Assume that $A \subset A^*$. From Proposition 7(1) we have $A^* \subset c_{\delta_\mu}(A)$. Suppose that $x \notin A^*$. From Proposition 8 we get $x\,\overline{\delta_\mu}A^*$. By hypothesis, $x\,\overline{\delta_\mu}A$. Thus, $x \notin c_{\delta_\mu}(A)$. So we have $c_{\delta_\mu}(A) \subset A^*$.
Sufficiency: Let $A^* = c_{\delta_\mu}(A)$ and $x \notin A^*$. Thus, $x \notin c_{\delta_\mu}(A)$ implies $x\ \overline{\delta_\mu}A$. Therefore, we have $x \ll_\mu X \setminus A$. Hence $\{x\} \subset X \setminus A$, that is $x \notin A$. □

Proposition 10. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. If A is δ_μ-closed, then $A^* \subset A$.

Proof. Let A be δ_μ-closed and $x \notin A$. Then $x \ll_\mu X\setminus A$. Since $A \cap (X \setminus A) = \emptyset \in \mathfrak{H}$, we have $x \notin A^*$. □

The following corollary follows from Proposition 8 and Proposition 10.

Corollary 1. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then $(A^*)^* \subset A^*$.

Theorem 3. Let (X, μ) be a normal generalised topological space and $A, B \subset X$. Then the relation δ_μ^n on X given by

$$A\delta_\mu^n B \text{ if and only if } c_\mu(A) \cap c_\mu(B) \neq \emptyset$$

defines a μ-proximity.

Proof.

(1) $A\delta_\mu^n B$ if and only if $B\delta_\mu^n A$.
(2) Suppose that $A\delta_\mu^n B, A \subset C$ and $B \subset D$. Since $A\delta_\mu^n B$, then $c_\mu(A) \cap c_\mu(B) \neq \emptyset$. This implies $c_\mu(C) \cap c_\mu(D) \neq \emptyset$. Thus, $C\delta_\mu^n D$.
(3) Since $\{x\} \subset c_\mu(\{x\})$ for each $x \in X$, we have $\{x\}\delta_\mu^n\{x\}$.
(4) Assume that $A\overline{\delta_\mu^n}B$. Thus, $c_\mu(A) \cap c_\mu(B) = \emptyset$. Since (X, μ) is a normal GTS, there exist two μ-open sets G and G' such that $c_\mu(A) \subset G, c_\mu(B) \subset G'$ and $G \cap G' = \emptyset$. Therefore, we have $c_\mu(A) \cap c_\mu(X\setminus G) = \emptyset$ and $c_\mu(B) \cap c_\mu(X\setminus G') = \emptyset$. If we take $E = G'$, then $A\overline{\delta_\mu^n}E$ and $B\overline{\delta_\mu^n}(X\setminus E)$. □

The following example shows that the converse implication of Proposition 10 may not be true in general.

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Example 3. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, $\mathfrak{H} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}\}$ and $A = \{a, b\}$. Consider μ-proximity δ_μ^n in Theorem 3. It is obvious that (X, μ) is normal GTS. Then $A^*(\delta_\mu^n) = \emptyset \subset A$ but A is not δ_μ^n-closed.

Proposition 11. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then $(A \cup A^*)^* \subset A^*$.

Proof. Let $x \notin A^*$. There exists a δ_μ-neighbourhood V of x such that $V \cap A \in \mathfrak{H}$. Then there exists a δ_μ-open set G containing x such that $G \subset V$. Then $G \cap A \in \mathfrak{H}$. By Lemma 3, $G \cap A^* = \emptyset$. Therefore, $G \cap (A \cup A^*) = G \cap A \in \mathfrak{H}$. Since G is also a δ_μ-neighbourhood of x, we have $x \notin (A \cup A^*)^*$. □

Theorem 4. Let (X, μ) be a μ-proximity space with strongly μ-codense hereditary class $\mathfrak{H}$ according to GT $\mu = \tau(\delta_\mu)$ and $\emptyset \neq A \subset X$. Then

(1) If A is δ_μ-open, $A \subset A^*$.

(2) If A is δ_μ-open, $A \notin \mathfrak{H}$.

Proof.

(1) Let A be δ_μ-open and $x \notin A^*$. Then there exists a δ_μ-neighbourhood V of x such that $V \cap A \in \mathfrak{H}$. V is also a $\tau(\delta_\mu)$-neighbourhood of x. Therefore, there exists a δ_μ-open set G containing x such that $G \subset V$. So we have $G \cap A \in \mathfrak{H}$. Since $\mathfrak{H}$ is strongly μ-codense, we obtain $G \cap A = \emptyset$ and since $x \in G$, it follows that $x \notin A$.

(2) Let A be δ_μ-open. Assume that $A \in \mathfrak{H}$. From (1) and Proposition 3 (2), we have $A = \emptyset$. This contradicts our hypothesis. So $A \notin \mathfrak{H}$. □

Theorem 4 may not be true in general if $\mathfrak{H}$ is not a strongly μ-codense hereditary class. The following example verifies this fact.

Example 4. Consider Example 2. $\mathfrak{H}$ is not a strongly μ-codense hereditary class according to GT $\mu = \tau(\delta) = \wp(X)$. Therefore, $A \not\subset A^* = \{a\}$ and $B \in \mathfrak{H}$ although $A = \{a, c\}$ and $B = \{c\}$ are δ-open.

The following corollary follows from Proposition 7(2) and Theorem 4(1).

Corollary 2. Let (X, δ_μ) be a μ-proximity space with strongly μ-codense hereditary class $\mathfrak{H}$ and A be a δ_μ-open subset of X. Then $x\ \delta_\mu A^*$ if and only if $x \delta_\mu A$.

NEW μ-PROXIMITY GENERATED BY $c^*_{\delta_\mu}$

In this last section we prove the main theorems on our new μ-proximity space.

Theorem 5. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then the operator $c^*_{\delta_\mu}(A) = A \cup A^*$ satisfies the following conditions.

(1) $A \subset c^*_{\delta_\mu}(A)$.

(2) $A \subset B$ implies $c^*_{\delta_\mu}(A) \subset c^*_{\delta_\mu}(B)$.

(3) $c^*_{\delta_\mu}(c^*_{\delta_\mu}(A)) = c^*_{\delta_\mu}(A)$.

(4) $c^*_{\delta_\mu}(A) \subset c_{\delta_\mu}(A)$.

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Proof.

(1) It is obvious.

(2) Let $A \subset B$. Then $c^*_{\delta_\mu}(A) = A \cup A^* \subset B \cup B^* = c^*_{\delta_\mu}(B)$.

(3) Since $A \subset c^*_{\delta_\mu}(A)$, we have $c^*_{\delta_\mu}(A) \subset c^*_{\delta_\mu}(c^*_{\delta_\mu}(A))$ by (2). For the other inclusion, we have $c^*_{\delta_\mu}\left(c^*_{\delta_\mu}(A)\right) = c^*_{\delta_\mu}(A \cup A^*) = (A \cup A^*) \cup (A \cup A^*)^* \subset (A \cup A^*) \cup A^* = c^*_{\delta_\mu}(A)$ by Proposition 11.

(4) Let $x \in c^*_{\delta_\mu}(A)$. If $x \in A$, then $x\delta_\mu A$ implies $x \in c_{\delta_\mu}(A)$. If $x \in A^*$, then $x \in c_{\delta_\mu}(A)$ from Proposition 7. □

The following remark follows from Lemma 2.

Remark 2. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A \subset X$. Then

(1) If $\mathfrak{H} = \{\emptyset\}$, $c^*_{\delta_\mu}(A) = c_{\delta_\mu}(A)$.

(2) If $\mathfrak{H} = \wp(X)$, $c^*_{\delta_\mu}(A) = A$.

Theorem 6. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A, B \subset X$. Then the relation δ^*_μ which is defined by

$$A\delta^*_\mu B \text{ if and only if } A \cap c^*_{\delta_\mu}(B) \neq \emptyset$$

is a quasi μ-proximity on X. Moreover, it is finer than δ_μ.

Proof.

(1) Let $A\delta^*_\mu B$, $A \subset C$ and $B \subset D$. Since $A\delta^*_\mu B$, we have $A \cap c^*_{\delta_\mu}(B) \neq \emptyset$. Now $A \subset C$ and $B \subset D$ imply $C \cap c^*_{\delta_\mu}(D) \neq \emptyset$. Thus, $C\delta^*_\mu D$.

(2) Since $\{x\} \subset c^*_{\delta_\mu}(\{x\})$, we obtain $\{x\} \cap c^*_{\delta_\mu}(\{x\}) = \{x\} \neq \emptyset$. Thus, $x\delta^*_\mu x$ for each $x \in X$.

(3) Let $A\,\overline{\delta^*_\mu}B$. Then $A \cap c^*_{\delta_\mu}(B) = \emptyset$. Assume $C = c^*_{\delta_\mu}(B)$. Thus, $A \cap c^*_{\delta_\mu}(C) = \emptyset$ implies $A\,\overline{\delta^*_\mu}C$. Also, $(X \setminus C) \cap c^*_{\delta_\mu}(B) = \emptyset$ implies $(X \setminus C)\overline{\delta^*_\mu}B$.

Hence δ^*_μ is a quasi μ-proximity. Let $A\delta^*_\mu B$. Then $A \cap c^*_{\delta_\mu}(B) \neq \emptyset$ implies $c_{\delta_\mu}(A) \cap c_{\delta_\mu}(B) \neq \emptyset$. Thus, we have $A\delta_\mu B$ by Lemma 1. So δ^*_μ is finer than δ_μ. □

Theorem 7. Let $(X, \delta_\mu, \mathfrak{H})$ be a hereditary μ-proximity space and $A, B \subset X$. Then the following hold.

(1) $A^*(\delta^*_\mu, \mathfrak{H}) \subset A^*(\delta_\mu, \mathfrak{H})$.

(2) If A is δ_μ- closed, it is also δ^*_μ-closed.

(3) $c^*_{\delta_\mu}(A) = c_{\delta^*_\mu}(A)$.

Proof.

(1) It is clear since $\delta_\mu < \delta^*_\mu$.

(2) Let A be δ_μ-closed and $x\delta^*_\mu A$. Since $\delta_\mu < \delta^*_\mu$, we have $x\delta_\mu A$. This implies by hypothesis that $x \in A$. Thus, A is also δ^*_μ-closed.

(3) Let $x \in c^*_{\delta_\mu}(A)$. Then we have $\{x\} \cap c^*_{\delta_\mu}(A) \neq \emptyset$. By Theorem 6, we get $x\delta^*_\mu A$. Thus, we obtain $x \in c_{\delta^*_\mu}(A)$. So $c^*_{\delta_\mu}(A) \subset c_{\delta^*_\mu}(A)$. The other inclusion is proved in a similar way. □

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The following example shows that the converse implication of Theorem 7(2) is not true.

Example 5. Consider Example 3. Then A is δ_μ^{n*}-closed but it is not δ_μ^n-closed.

REFERENCES

1. Á. Császár, "Generalized open sets", *Acta Math. Hungar.*, **1997**, *75*, 65-87.
2. Á. Császár, "Generalized topology, generalized continuity", *Acta Math. Hungar.*, **2002**, *96*, 351-357.
3. Á. Császár, "Modification of generalized topologies via hereditary classes", *Acta Math. Hungar.*, **2007**, *115*, 29-36.
4. C. Carpintero, E. Rosas, M. Salas-Brown and J. Sanabria, "μ-Compactness with respect to a hereditary class", *Bol. Soc. Paran. Mat.*, **2016**, *34*, 231-236.
5. V. Renukadevi and P. Vimaladevi, "Note on generalized topological spaces with hereditary classes", *Bol. Soc. Paran. Mat*., **2014**, *32*, 89-97.
6. A. Qahis and T. Noiri, "Functions and weakly $\mu\mathcal{H}$-compact spaces", *Europ. J. Pure Appl. Math.*, **2017**, *10*, 410-418.
7. V. A. Efremovič, "The geometry of a proximity", *Mat. Sb.*, **1952**, *31*, 189-200.
8. R. A. Hosny and O. A. E. Tantawy, "New proximities from old via ideals", *Acta Math. Hungar.*, **2006**, *110*, 37-50.
9. M. W. Lodato, "On topologically induced generalized proximity relations II", *Pasific J. Math*., **1966**, *17*, 131-135.
10. M. N. Mukherjee, D. Mandal and D. Dey, "Proximity structure on generalized topological spaces", *Afr. Mat.*, **2019**, *30*, 91-100.
11. S. A. Naimpally and B. D. Warrack, "Proximity Spaces", Cambridge University Press, Cambridge, **1970**.
12. Á. Császár, "Normal generalized topologies", *Acta Math. Hungar.*, **2007**, *115*, 309-313.