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Full Paper

Parameter estimation of Pareto distribution: some modified moment estimators

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Abstract: This article focuses on modified moment estimators for the parameter estimation of Pareto distribution. Four modified moment estimators are proposed and compared with the traditional method of moments estimation. The comparison is carried out through a simulation study using total relative deviation (TRD) and mean square error (MSE) as performance indicators. The simulation study was carried out using R-Language. Modified moment estimators based on *mean of first-order statistics* and *expectation of empirical cumulative distribution function of first-order statistics* are demonstrated to perform better than the traditional estimator and other modified moment estimators. The modified moment estimators perform much better when applied on real-life data where bootstrap MSE and TRD are used as performance measures.

Keywords: bootstrapping, method of moments, mean square error, modified moment estimators, Pareto distribution, total relative deviation

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INTRODUCTION

Pareto distribution was developed by Pareto [1] and it is widely used as an income model. The distribution was originally used to define the allocation of wealth among individual units since it seems to show rather well the way that a large proportion of wealth in any society is owned by a small percentage of people in that society [2]. According to Burroughs and Tebbens [3], the main applications of Pareto distribution are in the modelling of earthquakes, forestry fire areas, and oil and gas explorations in different field sizes. The importance of the distribution in real-life problems is evident in many studies [4-9].

Different estimation methods have been implemented to estimate the parameters of Pareto distribution. Quandt [10] derived algebraic expressions of different estimation methods, namely method of moments (MM), method of maximum likelihood, quantiles method and method of least squares. Afify [11] employed three distinct estimation procedures to estimate the parameters of Pareto distribution. He compared the maximum product spacing method of estimation with the ridge regression and least square methods. Lu and Tao [12] considered the weighted least-square method to estimate the parameters of Pareto distribution.

In the recent literature different modifications have been proposed in different estimators, which performed better than the traditional estimators in most cases. Such modifications have been applied for the parameter estimation of different probability distributions. The concept of a modified estimator was coined by Cohen and Whitten [13], who derived the modified moment estimators and modified maximum likelihood estimators for Weibull distribution. Mostly their modifications were based on order statistics. Their numerical evaluation showed that the modified moment and modified maximum likelihood estimators gave superior accuracy as compared to the traditional moment and maximum likelihood estimators respectively. Rashid and Akhter [14] have suggested some modifications in the MM and maximum likelihood for Exponential distribution. Their results are in favour of the modified estimators in comparison to the traditional ones, particularly in the case of modified moment estimators. Zaka and Akhter [15] have also proposed some modified moment and modified maximum likelihood estimators for Power Function distribution. Their results suggest that the modified estimators outperform the basic or traditional estimators. However, some of their proposed modifications derived through the MM have no practical meanings because two modified moment estimators have the same expressions when variance or coefficient of variation is replaced in the 2^{nd} moment of Power Function distribution. This makes the two modifications identical. These are the major drawbacks in their proposed modifications in the MM.

In this article we propose some modifications in the moment estimators for parameter estimation of Pareto distribution. These newly proposed modifications are derived using variance, harmonic mean, mean of first-order statistics and expectation of empirical cumulative distribution function (CDF) of first-order statistics. The core motivation behind all modifications is to estimate parameters of the widely applied Pareto distribution as efficiently and accurately as possible. The performance of the proposed modified estimators is compared through a simulation study and a real-life data example.

METHODS

We propose four modifications to the MM based on variance, harmonic mean, mean of first-order statistics and expectation of empirical CDF of first-order statistics of Pareto distribution. The following section briefly explains the derivation of moment estimators and proposed modified moment estimators for Pareto distribution.

Method of Moments (MM)

The MM proposed by Pearson [16] is one of the oldest methods for parameter estimation. Due to its conceptual and methodological simplicity, it is extensively used in the literature for parameter estimation of almost all probability distributions. This method consists in equating

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sample moments to corresponding moments of the parent population and then solving a system of equations for unknown parameters.

Consider $t_1, t_2, t_3, \ldots, t_n$ as a random sample from Pareto distribution with its probability density function given as

$$f(t) = \frac{\alpha\beta^{\alpha}}{t^{\alpha+1}} \qquad t \geq \beta \quad \text{and} \quad \alpha, \beta > 0 \cdot$$

The r^{th} sample moment is defined as $m_r' = \frac{\sum_{i=1}^{n} t_i^{\,r}}{n}$, where m_r' stands for r^{th} sample moment. The r^{th} population moment of Pareto distribution is defined as

$$E(t^r) = \int_{-\infty}^{+\infty} t^r . f(t)\, dt \ ,$$

$$E(t^r) = \frac{\alpha\beta^{\,r}}{\alpha - r} \cdot$$

Putting $\mathrm{r} = 1,\ 2$ we get the first two population moments about the origin denoted by μ_1' and μ_2':

$$\mu_1' = \frac{\alpha\beta}{\alpha - 1} \quad \text{and} \quad \mu_2' = \frac{\alpha\beta^2}{\alpha - 2} \cdot$$

Now equating the corresponding sample and population moments, we get

$$\mu_1' = m_1' \quad \text{and} \quad \mu_2' = m_2' \ ,$$

$$\frac{\alpha\beta}{\alpha - 1} = \bar{t} \ , \tag{1}$$

$$\frac{\alpha\beta^2}{\alpha - 2} = s^2 + \bar{t}^{\,2} \cdot \tag{2}$$

Solving equations (1) and (2), we get the estimates of α and β as

$$\hat{\alpha} = 1 + \sqrt{1 + \frac{\bar{t}^{\,2}}{s^2}} \ , \tag{3}$$

$$\hat{\beta} = \frac{\sqrt{s^2 + \bar{t}^{\,2}}}{s + \sqrt{s^2 + \bar{t}^{\,2}}}\,\bar{t} \ \cdot \tag{4}$$

Equations (3) and (4) are the estimators derived through MM for Pareto distribution.

Modified Moment Estimator (I)

For the first modification in the MM, we followed Cohen and Whitten [13], Rashid and Akhter [14] and Zaka and Akhter [15], who derived the modified moment estimators for Weibull, Exponential and Power Function distributions respectively. In this modification which is based on variance, equation (2) is replaced by the variance of Pareto distribution. The detailed derivation is given as

$$s^2 = \frac{\alpha\beta^2}{(\alpha - 1)^2 . (\alpha - 2)} \ \cdot \tag{5}$$

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From equation (5) we get

$$\beta = s\left(\alpha - 1\right).\sqrt{1 - \frac{2}{\alpha}} \cdot \tag{6}$$

Solving equations (1) and (6) for α and β, we get the estimator for unknown parameters as

$$\hat{\alpha} = 1 + \sqrt{1 + \frac{\bar{t}^2}{s^2}} \cdot \tag{7}$$

Putting the value of $\hat{\alpha}$ from equation (7) in equation (6), we get

$$\beta = s\left(1 + \sqrt{\left(1 + \frac{\bar{t}^2}{s^2}\right)} - 1\right).\left(\sqrt{1 - \frac{2}{1 + \sqrt{\left(1 + \frac{\bar{t}^2}{s^2}\right)}}}\right),$$

$$= \left(\sqrt{s^2 + \bar{t}^2}\right)\sqrt{\frac{s + \sqrt{s^2 + \bar{t}^2} - 2s}{s + \sqrt{s^2 + \bar{t}^2}}},$$

$$\hat{\beta} = \sqrt{\left(s^2 + \bar{t}^2\right)\left(\frac{\sqrt{s^2 + \bar{t}^2} - s}{s + \sqrt{s^2 + \bar{t}^2}}\right)} \cdot \tag{8}$$

Equations (7) and (8) provide algebraic expressions for the 1st modified moment (MM-I) estimators of Pareto distribution.

Modified Moment Estimator (II)

Our second modification is based on the harmonic mean of Pareto distribution. For this we also followed the pattern used by Cohen and Whitten [13], Rashid and Akhter [14] and Zaka and Akhter [15], for modified estimators of Weibull, Exponential and Power Function distributions respectively. In this modification equation (2) is replaced by the harmonic mean (HM) of Pareto distribution:

$$HM = \beta\left(1 + \frac{1}{\alpha}\right) \cdot \tag{9}$$

From equation (9),

$$\beta = \frac{HM}{\left(1 + \frac{1}{\alpha}\right)} \cdot \tag{10}$$

Solving equations (3) and (10) simultaneously for unknown parameters, we get the required estimators of α and β as

$$\hat{\alpha} = \sqrt{1 + \frac{HM}{\bar{t}}}, \tag{11}$$

$$\hat{\beta} = \frac{\sqrt{\bar{t} + H.M}\left(HM\right)}{\sqrt{\bar{t}} + \sqrt{\left(\bar{t} + HM\right)}} \cdot \tag{12}$$

Thus, equations (11) and (12) provide algebraic expressions of the 2nd modified moment (MM-II) estimators of Pareto distribution.

Modified Moment Estimator (III)

For the third modification in the MM, we followed the procedure given by Rashid and Akhter [14], who derived the modified moment estimator for Exponential distribution. In this modification the second moment about origin (i.e. equation 2) is replaced by the mean of first-order statistics of Pareto distribution. The detailed derivation is given below. The first moment about origin of Pareto distribution is

$$\frac{\alpha\beta}{\alpha-1}=\bar{t}\ . \tag{13}$$

In general, for any probability distribution the mean of first order-statistics is

$$E\left(t_{(1)}\right)=t_{(1)}\ .$$

For the particular case of Pareto distribution, the mean of fist-order statistics from Afify [17] is

$$\frac{\alpha\beta n}{\alpha n-1}=t_{(1)}\ . \tag{14}$$

Solving equations (13) and (14), we get

$$\hat{\beta}=t_{(1)}-\frac{t_{(1)}}{n\alpha}\ , \tag{15}$$

$$\hat{\alpha}=\frac{t_{(1)}-n\bar{t}}{n\left(t_{(1)}-\bar{t}\right)}\ . \tag{16}$$

Equations (15) and (16) are the 3rd modified moment (MM-III) estimators of Pareto distribution.

Modified Moment Estimator (IV)

For our fourth modification in the MM, we followed Rashid and Akhter [14], who derived the modified moment estimator for Exponential distribution. In this modification the 2nd moment about origin (i.e. equation 2) is replaced by expectation of the empirical CDF of first-order statistics.

From equation (1), we have

$$\frac{\alpha}{\alpha-1}=\frac{\bar{t}}{\beta}\ . \tag{17}$$

From the literature [13-15], the expectation of empirical CDF of first-order statistics is

$$E\left[F\left(t_{(1)}\right)\right]=F\left(t_{(1)}\right)=\frac{1}{n+1}\ , \tag{18}$$

$$\frac{1}{n+1}=1-\left(\frac{\beta}{t_{(1)}}\right)^{\alpha}\ ,$$

$$\hat{\beta}=t_{(1)}\left(\frac{n}{n+1}\right)^{\left(\frac{1}{\alpha}\right)}\ . \tag{19}$$

Putting the value of β from equation (19) in equation (17), we get

$$\frac{t_{(1)}}{\bar{t}} = \left(1+\frac{1}{n}\right)^{\frac{1}{\alpha}}\left(1-\frac{1}{\alpha}\right).$$

Applying Binomial series on the first term of the right hand side and ignoring the higher-order terms,

$$\frac{t_{(1)}}{\bar{t}} = \left(1-\frac{1}{\alpha}+\frac{1}{\alpha n}\right),$$

$$\hat{\alpha} = \frac{(n-1)\bar{t}}{n\left(\bar{t}-t_{(1)}\right)}. \qquad (20)$$

Equations (19) and (20) are algebraic expressions for the 4th modified moment (MM-IV) estimators of Pareto distribution.

Simulation Study

A simulation study was performed to compare the performance of the proposed modified moment estimators with one another and with traditional moment estimators. The Simulation study was carried out for eight different combinations of parameter values ($\alpha=2,\beta=3$; $\alpha=3,\beta=4$; $\alpha=0.5,\beta=1$; $\alpha=1,\beta=1$; $\alpha=2,\beta=1$; $\alpha=4,\beta=1$; $\alpha=1,\beta=2$; and $\alpha=4,\beta=5$) with various sample sizes (n = 20, 50, 100, 200, 500 and 1000). We generated random samples of different sizes by observing that if R is uniform (0, 1), then $t_i = \frac{\beta}{(1-R)^{\frac{1}{\alpha}}}$ is the Pareto random variable with parameters α and β. All the simulation results were based on 10,000 replications using R-Language [18]. Total relative deviation (TRD) and mean square error (MSE) were used as performance indicators, which are defined as

$$TRD = \left|\frac{\hat{\alpha}-\alpha}{\alpha}\right| + \left|\frac{\hat{\beta}-\beta}{\beta}\right|,$$

$$MSE_{\hat{\alpha}} = E\left(\hat{\alpha}-\alpha\right)^2,$$

$$MSE_{\hat{\beta}} = E\left(\hat{\beta}-\beta\right)^2.$$

These performance indicators are commonly used for this purpose in such type of comparisons as in Al-Fawzan [19], Rashid and Akhter [14] and Zaka and Akhter [15].

RESULTS AND DISCUSSION

Results from Simulation Study

Results from the simulation study are given in Tables 1-8 for different combinations of parameter values. These results were obtained for different sample sizes and used for analysing the accuracy of estimators derived through the MM and the proposed modified estimators.

From the results in Table 1 $(\text{for}\,\beta=3\,,\alpha=2)$, it is observed that for scale parameter (β), MM and MM-I overestimate while MM-II underestimates the true parameter values. The estimates from MM-III and MM-IV are much closer to the true parameter values. Similarly, with respect to shape parameter (α), MM and MM-I overestimate while MM-II underestimates the true parameter values. However, estimates from MM-III and MM-IV are much closer to the true values. In general, MM-III and MM-IV outperform the traditional moment estimator and other two modified moment estimators throughout the sample sizes based on both of the performance indices used (i.e. MSE and TRD). For example, for first parameter combination $(\beta = 3, \alpha = 2)$, the values of TRD for MM-IV (TRD = 0.092784, 0.047682, 0.026854, 0.017388, 0.008749 and 0.004681 for sample size 20, 50, 100, 200, 500 and 1000 respectively) are much lower than those for MM, MM-I and MM-II, and closer (but lower) than that for MM-III.

Similarly, values of MSE for different sample sizes are lower in the case of MM-IV compared to other estimators considered. Similar patterns of results are evident in Tables 2-8 for other parameter combinations considered.

Although MM-III and MM-IV have almost identical results, we can make a choice in favour of MM-IV based on the values of both performance indices, while MM and MM-I are found to be poorest performers, particularly for small samples.

Application to Real-Life Data

After comparing the modified moment estimators by means of simulation study, we compared the performance of these estimators using a set of real-life data taken from Beirliant et al. [20]. The data consisted of fire damage claims (in thousands of Norwegian Krones) in Norway during 1975. The same data have also been used in different studies on Pareto distribution like those by Munir et al. [2], Brazauskas and Serfling [21], Kaiser and Brazauskas [22], and Obradović [23]. First, it was determined that the data followed Pareto distribution by applying Kolmogorov-Smirnov test (p-value>0.75) [24-25]. In comparison based on real-life data, bootstrap estimates of bias, mean square error and total relative deviation were used as performance indicators. Bootstrapping [26-28] was performed using ‘boot’ package in R-language [29] with 1000 bootstrap replicates. Specifically, the following formulas were used for calculations of estimated bias, estimated MSE and estimated TRD with bootstrap samples:

$$Estimated\ Bias(\hat{\alpha}) = \mathrm{E}(\hat{\alpha}_{\substack{Bootstrapp\\Samples}}) - \hat{\alpha}_{\substack{Original\\Sample}} ,$$

$$Estimated\ Bias(\hat{\beta}) = \mathrm{E}(\hat{\beta}_{\substack{Bootstrapp\\Samples}}) - \hat{\beta}_{\substack{Original\\Sample}} ,$$

$$MSE(\hat{\alpha}) = \mathrm{E}\left(\hat{\alpha}_{\substack{Bootstrapp\\Samples}} - \hat{\alpha}_{\substack{Original\\Sample}}\right)^2 ,$$

$$MSE(\hat{\beta}) = \mathrm{E}\left(\hat{\beta}_{\substack{Bootstrapp\\Samples}} - \hat{\beta}_{\substack{Original\\Sample}}\right)^2 ,$$

$$TRD = \frac{\left|\mathrm{E}(\hat{\alpha}_{\substack{Bootstrapp\\Samples}}) - \hat{\alpha}_{\substack{Original\\Sample}}\right|}{\hat{\alpha}_{\substack{Original\\Sample}}} + \frac{\left|\mathrm{E}(\hat{\beta}_{\substack{Bootstrapp\\Samples}}) - \hat{\beta}_{\substack{Original\\Sample}}\right|}{\hat{\beta}_{\substack{Original\\Sample}}} .$$

Results from the real-life data are given in Table 9. These results show that MM-III and MM-IV perform much better than other competing estimators. However, the results from these

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two estimators are almost identical in terms of bias, MSE and TRD computed through bootstrapping.

Table 1. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 2,\ \beta = 3$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	3	2	3.865413	3.060355	1.893612	1.59739	0.818649
MM-I	20	3	2	3.876016	3.051628	1.992801	1.575232	0.817819
MM-II	20	3	2	2.586786	1.336966	0.210114	0.441711	0.469255
MM-III	20	3	2	3.004632	2.225875	0.006413	0.311636	0.114481
MM-IV	20	3	2	3.005373	2.181986	0.006374	0.291313	0.092784
MM	50	3	2	3.714096	2.7445	0.772574	0.736287	0.610282
MM-I	50	3	2	3.70106	2.740219	0.730394	0.732384	0.603796
MM-II	50	3	2	2.575019	1.32981	0.195976	0.450497	0.476755
MM-III	50	3	2	3.000896	2.107677	0.000934	0.113145	0.054137
MM-IV	50	3	2	3.00121	2.094557	0.00093	0.113097	0.047682
MM	100	3	2	3.629124	2.599405	0.612961	0.459771	0.509411
MM-I	100	3	2	3.632687	2.60072	0.665045	0.462434	0.511255
MM-II	100	3	2	2.569088	1.32777	0.193439	0.452722	0.479753
MM-III	100	3	2	3.000277	2.059437	0.000236	0.060188	0.029811
MM-IV	100	3	2	3.000257	2.053537	0.00023	0.05945	0.026854
MM	200	3	2	3.552845	2.493836	0.375271	0.305276	0.4312
MM-I	200	3	2	3.557561	2.496145	0.532833	0.308175	0.433926
MM-II	200	3	2	2.566255	1.325217	0.191955	0.455881	0.481973
MM-III	200	3	2	3.000046	2.037499	5.50E-05	0.032376	0.018765
MM-IV	200	3	2	3.000099	2.034711	5.65E-05	0.031908	0.017388
MM	500	3	2	3.480513	2.401678	0.319228	0.195656	0.36101
MM-I	500	3	2	3.477167	2.40179	0.302164	0.196574	0.359951
MM-II	500	3	2	2.564191	1.32422	0.191459	0.456957	0.483159
MM-III	500	3	2	3.000022	2.01751	8.89E-06	0.015523	0.008762
MM-IV	500	3	2	3.000011	2.017491	9.09E-06	0.014832	0.008749
MM	1000	3	2	3.428693	2.348959	0.199809	0.145397	0.317377
MM-I	1000	3	2	3.428557	2.351062	0.201121	0.146736	0.318383
MM-II	1000	3	2	2.564052	1.323799	0.190832	0.457406	0.483417
MM-III	1000	3	2	2.999999	2.011276	2.23E-06	0.008477	0.005638
MM-IV	1000	3	2	3.000021	2.009348	2.28E-06	0.008236	0.004681

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Table 2. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 3,\ \beta = 4$

Method	Sample Size	True β	True α	$E(\hat{\beta})$	$E(\hat{\alpha})$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	4	3	4.37002	4.044718	0.22313	2.299626	0.440744
MM-I	20	4	3	4.375302	4.043572	0.229597	2.288481	0.441683
MM-II	20	4	3	3.09911	1.377586	0.841095	2.632762	0.766027
MM-III	20	4	3	4.000532	3.267615	0.004717	0.690579	0.089338
MM-IV	20	4	3	4.004029	3.202262	0.005051	0.648721	0.068428
MM	50	4	3	4.256702	3.613125	0.109295	0.862285	0.26855
MM-I	50	4	3	4.258372	3.611925	0.119125	0.857282	0.268568
MM-II	50	4	3	3.091999	1.375833	0.835764	2.63819	0.768389
MM-III	50	4	3	4.000619	3.101963	0.000752	0.228561	0.034142
MM-IV	50	4	3	4.00055	3.085907	0.000733	0.229814	0.028773
MM	100	4	3	4.190824	3.424327	0.069547	0.466526	0.189148
MM-I	100	4	3	4.195005	3.430465	0.071769	0.472042	0.19224
MM-II	100	4	3	3.089675	1.375235	0.83444	2.640012	0.76917
MM-III	100	4	3	4.000116	3.05568	0.000187	0.112585	0.018589
MM-IV	100	4	3	4.00012	3.046318	0.00018	0.112386	0.015469
MM	200	4	3	4.14393	3.297639	0.048115	0.260341	0.135195
MM-I	200	4	3	4.14444	3.301161	0.048371	0.266786	0.136497
MM-II	200	4	3	3.088585	1.374612	0.833421	2.64197	0.76965
MM-III	200	4	3	4.000147	3.027472	4.76E-05	0.056935	0.009194
MM-IV	200	4	3	3.999987	3.018098	4.47E-05	0.056043	0.006036
MM	500	4	3	4.093546	3.18984	0.032568	0.140336	0.086667
MM-I	500	4	3	4.099083	3.193013	0.0317	0.137509	0.089108
MM-II	500	4	3	3.088541	1.374392	0.831877	2.642638	0.769734
MM-III	500	4	3	4.000022	3.012012	7.22E-06	0.023447	0.004009
MM-IV	500	4	3	4.000046	3.006294	7.40E-06	0.022965	0.00211
MM	1000	4	3	4.07309	3.143341	0.024186	0.089499	0.066053
MM-I	1000	4	3	4.073819	3.141556	0.024692	0.089071	0.06564
MM-II	1000	4	3	3.086933	1.374543	0.834257	2.642127	0.770086
MM-III	1000	4	3	4.000023	3.007913	1.76E-06	0.011486	0.002643
MM-IV	1000	4	3	4.000005	3.006556	1.80E-06	0.011672	0.002186

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Table 3. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 0.5,\ \beta = 1$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	1	0.5	188742.6	2.082543	1.05E+14	2.510318	188744.8
MM-I	20	1	0.5	188742.6	2.082543	1.05E+14	2.510318	188744.8
MM-II	20	1	0.5	1.596848	1.040831	0.478534	0.29489	1.67851
MM-III	20	1	0.5	1.057601	1.034635	0.017835	0.288489	1.126871
MM-IV	20	1	0.5	1.057575	0.984635	0.017832	0.237526	1.026846
MM	50	1	0.5	108408.6	2.03216	1.11E+13	2.348319	108410.7
MM-I	50	1	0.5	108408.6	2.03216	1.11E+13	2.348319	108410.7
MM-II	50	1	0.5	1.535923	1.017739	0.327199	0.268592	1.571401
MM-III	50	1	0.5	1.021114	1.013155	0.002276	0.263696	1.047425
MM-IV	50	1	0.5	1.021113	0.993155	0.002276	0.24357	1.007423
MM	100	1	0.5	1058666	2.016096	3.18E+15	2.298747	1058668
MM-I	100	1	0.5	1058666	2.016096	3.18E+15	2.298747	1058668
MM-II	100	1	0.5	1.518801	1.009247	0.287691	0.259495	1.537294
MM-III	100	1	0.5	1.009998	1.006515	0.000504	0.256648	1.023028
MM-IV	100	1	0.5	1.009998	0.996515	0.000504	0.246618	1.003028
MM	200	1	0.5	2510846	2.007934	2.25E+16	2.27391	2510848
MM-I	200	1	0.5	2510846	2.007934	2.25E+16	2.27391	2510848
MM-II	200	1	0.5	1.509976	1.004656	0.269323	0.254719	1.519287
MM-III	200	1	0.5	1.005147	1.00319	0.000129	0.253221	1.011527
MM-IV	200	1	0.5	1.005147	0.99819	0.000129	0.248214	1.001527
MM	500	1	0.5	34923570	2.003189	7.97E+18	2.259586	34923572
MM-I	500	1	0.5	34923570	2.003189	7.97E+18	2.259586	34923572
MM-II	500	1	0.5	1.504751	1.001878	0.258466	0.251888	1.508506
MM-III	500	1	0.5	1.002035	1.001263	2.07E-05	0.251268	1.004562
MM-IV	500	1	0.5	1.002035	0.999263	2.07E-05	0.249267	1.000562
MM	1000	1	0.5	8558908	2.001605	1.37E+17	2.254821	8558910
MM-I	1000	1	0.5	8558908	2.001605	1.37E+17	2.254821	8558910
MM-II	1000	1	0.5	1.501723	1.000964	0.253559	0.250966	1.503651
MM-III	1000	1	0.5	1.001021	1.000646	5.21E-06	0.250647	1.002312
MM-IV	1000	1	0.5	1.001021	0.999646	5.21E-06	0.249647	1.000312

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Table 4. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 1$, $\beta = 1$

Method	Sample Size	True β	True α	$E(\hat{\beta})$	$E(\hat{\alpha})$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	1	1	6.323339	2.282167	1736.812	1.706302	6.605506
MM-I	20	1	1	6.323339	2.282167	1736.812	1.706302	6.605506
MM-II	20	1	1	1.103215	1.191149	0.031866	0.043602	0.294364
MM-III	20	1	1	1.011386	1.313426	0.003072	0.146088	0.324812
MM-IV	20	1	1	1.01155	1.263426	0.003078	0.117245	0.274976
MM	50	1	1	7.059634	2.145868	3632.459	1.329753	7.205502
MM-I	50	1	1	7.059634	2.145868	3632.459	1.329753	7.205502
MM-II	50	1	1	1.079657	1.160847	0.014071	0.030755	0.240504
MM-III	50	1	1	1.003954	1.231872	0.000477	0.072548	0.235826
MM-IV	50	1	1	1.00398	1.211872	0.000478	0.063673	0.215852
MM	100	1	1	7.208215	2.085305	3423.682	1.183724	7.29352
MM-I	100	1	1	7.208215	2.085305	3423.682	1.183724	7.29352
MM-II	100	1	1	1.069045	1.14222	0.00869	0.023682	0.211264
MM-III	100	1	1	1.001361	1.191829	0.000101	0.046949	0.19319
MM-IV	100	1	1	1.001367	1.181829	0.000101	0.043213	0.183196
MM	200	1	1	9.625644	2.053104	39849.33	1.111338	9.678748
MM-I	200	1	1	9.625644	2.053104	39849.33	1.111338	9.678748
MM-II	200	1	1	1.06162	1.129475	0.005949	0.019339	0.191096
MM-III	200	1	1	1.000815	1.167775	2.74E-05	0.03448	0.16859
MM-IV	200	1	1	1.000816	1.162775	2.74E-05	0.032827	0.163591
MM	500	1	1	7.641622	2.026591	1167.722	1.054511	7.668214
MM-I	500	1	1	7.641622	2.026591	1167.722	1.054511	7.668214
MM-II	500	1	1	1.054059	1.113156	0.003929	0.014602	0.167215
MM-III	500	1	1	1.000226	1.140413	4.18E-06	0.023417	0.140639
MM-IV	500	1	1	1.000226	1.138413	4.18E-06	0.02286	0.138639
MM	1000	1	1	11.08374	2.015884	16812.18	1.032262	11.09962
MM-I	1000	1	1	11.08374	2.015884	16812.18	1.032262	11.09962
MM-II	1000	1	1	1.049414	1.103903	0.003055	0.012194	0.153318
MM-III	1000	1	1	1.0001	1.126045	1.03E-06	0.018523	0.126145
MM-IV	1000	1	1	1.0001	1.125045	1.03E-06	0.018271	0.125146

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Table 5. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 2,\ \beta = 1$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	1	2	1.284047	3.049478	0.160559	1.573503	0.808786
MM-I	20	1	2	1.284047	3.049478	0.160559	1.573503	0.808786
MM-II	20	1	2	0.863138	1.336502	0.023125	0.442285	0.468611
MM-III	20	1	2	1.00137	2.232215	0.000677	0.322007	0.117477
MM-IV	20	1	2	1.001653	2.182215	0.000678	0.301285	0.09276
MM	50	1	2	1.236925	2.743602	0.099953	0.731757	0.608726
MM-I	50	1	2	1.236925	2.743602	0.099953	0.731757	0.608726
MM-II	50	1	2	0.858502	1.330463	0.021701	0.449517	0.476266
MM-III	50	1	2	1.000217	2.111108	0.000103	0.113315	0.055771
MM-IV	50	1	2	1.000265	2.091108	0.000103	0.10927	0.045819
MM	100	1	2	1.208051	2.592582	0.058451	0.45177	0.504342
MM-I	100	1	2	1.208051	2.592582	0.058451	0.45177	0.504342
MM-II	100	1	2	0.85621	1.326774	0.021497	0.454074	0.480403
MM-III	100	1	2	1.000082	2.060213	2.58E-05	0.058424	0.030189
MM-IV	100	1	2	1.000094	2.050213	2.58E-05	0.05732	0.025201
MM	200	1	2	1.183251	2.498924	0.039025	0.311556	0.432713
MM-I	200	1	2	1.183251	2.498924	0.039025	0.311556	0.432713
MM-II	200	1	2	0.855622	1.325633	0.021264	0.455271	0.481561
MM-III	200	1	2	1.000054	2.037767	6.36E-06	0.032334	0.018938
MM-IV	200	1	2	1.000057	2.032767	6.36E-06	0.031981	0.016441
MM	500	1	2	1.158827	2.401757	0.027884	0.196476	0.359706
MM-I	500	1	2	1.158827	2.401757	0.027884	0.196476	0.359706
MM-II	500	1	2	0.854856	1.32398	0.021236	0.457273	0.483154
MM-III	500	1	2	0.999999	2.016268	9.85E-07	0.014875	0.008135
MM-IV	500	1	2	0.999999	2.014268	9.85E-07	0.014814	0.007135
MM	1000	1	2	1.143096	2.349746	0.023157	0.145729	0.317969
MM-I	1000	1	2	1.143096	2.349746	0.023157	0.145729	0.317969
MM-II	1000	1	2	0.854538	1.323536	0.021246	0.457765	0.483694
MM-III	1000	1	2	1.000001	2.009641	2.43E-07	0.008388	0.004822
MM-IV	1000	1	2	1.000001	2.008641	2.43E-07	0.00837	0.004322

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Table 6. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 4,\ \beta = 1$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	1	4	1.045923	5.11538	0.004576	3.374034	0.324768
MM-I	20	1	4	1.045923	5.11538	0.004576	3.374034	0.324768
MM-II	20	1	4	0.729034	1.393713	0.074452	6.792898	0.922538
MM-III	20	1	4	1.000202	4.305117	0.00017	1.146976	0.076482
MM-IV	20	1	4	1.000422	4.255117	0.00017	1.118964	0.064201
MM	50	1	4	1.027571	4.58218	0.002502	1.219718	0.173116
MM-I	50	1	4	1.027571	4.58218	0.002502	1.219718	0.173116
MM-II	50	1	4	0.728152	1.392416	0.074319	6.799583	0.923744
MM-III	50	1	4	1.00012	4.115159	2.72E-05	0.395033	0.028909
MM-IV	50	1	4	1.000156	4.095159	2.72E-05	0.390827	0.023946
MM	100	1	4	1.018613	4.371128	0.001788	0.633997	0.111395
MM-I	100	1	4	1.018613	4.371128	0.001788	0.633997	0.111395
MM-II	100	1	4	0.727889	1.392232	0.074245	6.800498	0.924053
MM-III	100	1	4	1.000007	4.054672	6.55E-06	0.179834	0.013675
MM-IV	100	1	4	1.000017	4.044672	6.55E-06	0.178841	0.011184
MM	200	1	4	1.012541	4.240264	0.001251	0.357779	0.072607
MM-I	200	1	4	1.012541	4.240264	0.001251	0.357779	0.072607
MM-II	200	1	4	0.727693	1.392083	0.074254	6.801252	0.924286
MM-III	200	1	4	1.000001	4.027886	1.56E-06	0.093466	0.006972
MM-IV	200	1	4	1.000003	4.022886	1.56E-06	0.093212	0.005725
MM	500	1	4	1.006529	4.122616	0.000774	0.172417	0.037183
MM-I	500	1	4	1.006529	4.122616	0.000774	0.172417	0.037183
MM-II	500	1	4	0.727515	1.391948	0.074288	6.801946	0.924498
MM-III	500	1	4	1.000001	4.009044	2.57E-07	0.035928	0.002262
MM-IV	500	1	4	1.000002	4.007044	2.57E-07	0.035896	0.001763
MM	1000	1	4	1.004619	4.087168	0.000509	0.104557	0.026411
MM-I	1000	1	4	1.004619	4.087168	0.000509	0.104557	0.026411
MM-II	1000	1	4	0.727399	1.392026	0.074331	6.801533	0.924595
MM-III	1000	1	4	0.999998	4.008756	6.07E-08	0.017958	0.002191
MM-IV	1000	1	4	0.999998	4.007756	6.07E-08	0.017941	0.001941

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Table 7. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 1,\ \beta = 2$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	2	1	12.1902	2.28732	11288.02	1.722971	6.382422
MM-I	20	2	1	12.1902	2.28732	11288.02	1.722971	6.382422
MM-II	20	2	1	2.204576	1.191697	0.121577	0.043931	0.293985
MM-III	20	2	1	2.022067	1.315383	0.011578	0.148422	0.326416
MM-IV	20	2	1	2.022395	1.265383	0.011601	0.119384	0.27658
MM	50	2	1	14.95611	2.143787	13191.63	1.324648	7.621845
MM-I	50	2	1	14.95611	2.143787	13191.63	1.324648	7.621845
MM-II	50	2	1	2.161444	1.161179	0.057609	0.030773	0.241901
MM-III	50	2	1	2.007157	1.231844	0.001756	0.072171	0.235423
MM-IV	50	2	1	2.007208	1.211844	0.001757	0.063298	0.215448
MM	100	2	1	17.23087	2.085	105864.8	1.183075	8.700434
MM-I	100	2	1	17.23087	2.085	105864.8	1.183075	8.700434
MM-II	100	2	1	2.138943	1.142029	0.035537	0.023659	0.211501
MM-III	100	2	1	2.003183	1.191452	0.000432	0.046836	0.193044
MM-IV	100	2	1	2.003195	1.181452	0.000432	0.043106	0.18305
MM	200	2	1	17.34105	2.052328	33657.19	1.109647	8.722852
MM-I	200	2	1	17.34105	2.052328	33657.19	1.109647	8.722852
MM-II	200	2	1	2.124067	1.128546	0.023862	0.019084	0.19058
MM-III	200	2	1	2.001508	1.166058	0.000102	0.0338	0.166812
MM-IV	200	2	1	2.001511	1.161058	0.000102	0.032165	0.161813
MM	500	2	1	17.62144	2.026676	31668.94	1.05467	8.837398
MM-I	500	2	1	17.62144	2.026676	31668.94	1.05467	8.837398
MM-II	500	2	1	2.108734	1.114023	0.015702	0.01474	0.16839
MM-III	500	2	1	2.000498	1.141502	1.64E-05	0.023598	0.141751
MM-IV	500	2	1	2.000498	1.139502	1.64E-05	0.023036	0.139751
MM	1000	2	1	14.20793	2.016202	2235.708	1.032915	7.120166
MM-I	1000	2	1	14.20793	2.016202	2235.708	1.032915	7.120166
MM-II	1000	2	1	2.099687	1.104949	0.012332	0.012366	0.154792
MM-III	1000	2	1	2.000188	1.127445	3.84E-06	0.018816	0.127539
MM-IV	1000	2	1	2.000188	1.126445	3.84E-06	0.018563	0.126539

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Table 8. Comparison of MM, MM-I, MM-II, MM-III and MM-IV for $\alpha = 4,\ \beta = 5$

Method	Sample Size	True β	True α	$E\left(\hat{\beta}\right)$	$E\left(\hat{\alpha}\right)$	MSE ($\hat{\beta}$)	MSE ($\hat{\alpha}$)	TRD
MM	20	5	4	5.226018	5.093195	0.113751	3.316351	0.318502
MM-I	20	5	4	5.226018	5.093195	0.113751	3.316351	0.318502
MM-II	20	5	4	3.645326	1.393485	1.861527	6.794091	0.922563
MM-III	20	5	4	5.000659	4.299591	0.004249	1.151293	0.07503
MM-IV	20	5	4	5.001759	4.249591	0.004249	1.123834	0.06275
MM	50	5	4	5.138879	4.576552	0.062943	1.188785	0.171914
MM-I	50	5	4	5.138879	4.576552	0.062943	1.188785	0.171914
MM-II	50	5	4	3.64204	1.392477	1.854058	6.799258	0.923473
MM-III	50	5	4	5.000482	4.1075	0.000668	0.376672	0.026971
MM-IV	50	5	4	5.000665	4.0875	0.000668	0.372772	0.022008
MM	100	5	4	5.094468	4.380518	0.04302	0.633311	0.114023
MM-I	100	5	4	5.094468	4.380518	0.04302	0.633311	0.114023
MM-II	100	5	4	3.638349	1.392324	1.859053	6.800016	0.924249
MM-III	100	5	4	5.000015	4.062338	0.000161	0.183676	0.015587
MM-IV	100	5	4	5.000061	4.052338	0.000161	0.182529	0.013097
MM	200	5	4	5.061062	4.233977	0.030192	0.346986	0.070707
MM-I	200	5	4	5.061062	4.233977	0.030192	0.346986	0.070707
MM-II	200	5	4	3.638101	1.39208	1.857259	6.801266	0.92436
MM-III	200	5	4	5.000001	4.027606	3.81E-05	0.088759	0.006902
MM-IV	200	5	4	5.000013	4.022606	3.81E-05	0.088508	0.005654
MM	500	5	4	5.034063	4.127541	0.018916	0.169143	0.038698
MM-I	500	5	4	5.034063	4.127541	0.018916	0.169143	0.038698
MM-II	500	5	4	3.637374	1.391994	1.857743	6.801706	0.924527
MM-III	500	5	4	5.00004	4.011459	6.22E-06	0.035424	0.002873
MM-IV	500	5	4	5.000042	4.009459	6.22E-06	0.035382	0.002373
MM	1000	5	4	5.021814	4.07965	0.012414	0.101932	0.024275
MM-I	1000	5	4	5.021814	4.07965	0.012414	0.101932	0.024275
MM-II	1000	5	4	3.637344	1.391963	1.857336	6.80186	0.92454
MM-III	1000	5	4	5.000009	4.005262	1.64E-06	0.018161	0.001317
MM-IV	1000	5	4	5.00001	4.004262	1.64E-06	0.018152	0.001067

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Table 9. Application of proposed estimators to real-life data

Method	Parameter	Estimate	Estimated Bias	Estimated MSE	Estimated TRD
MM	α	2.007348	0.082401	0.020149	0.566413
	β	2685.672	151.838	3606667	
MM-I	α	2.007348	0.082401	0.020149	0.566413
	β	2685.672	151.838	3606667	
MM-II	α	1.081603	0.046057	0.009887	0.108227
	β	472.3568	9.661112	724.1127	
MM-III	α	1.10233	0.102974	0.042571	0.122851
	β	496.8057	0.663232	6.640452	
MM-IV	α	1.095288	0.102974	0.042571	0.123632
	β	496.8067	0.663648	6.641755	

CONCLUSIONS

Four modifications in the method of moments for parameter estimation of Pareto distribution have been presented. Through a simulation study it has been established that the modified estimators based on *mean of first-order statistics* and *expectation of empirical cumulative distribution function of first-order statistics* are found superior to the traditional and modified moment estimators derived using *variance* and *harmonic mean*. These two modified estimators may be recommended for parameter estimation as they minimise the magnitude of underestimation or overestimation of parameters.

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